Stochastic Choice and Optimal Sequential Sampling

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Suppose we see an agent's choices from the menu \{a,b\}. Sometimes she chooses a, sometimes she chooses b. How to model?

- *Choice correspondence*: the agent is indifferent, regardless of the relative probabilities.

- *Stochastic choice*: treat the choice probabilities as data.

- *This talk*: Data is the joint distribution of choice probabilities and decision times.

*Note*: data on decision times has been used to classify decisions as "instinctive/heuristic" or "cognitive" by e.g. Rubinstein (2007), Rand et al (2012), Caplin and Dean (2014) -not our focus here.
Other related work


Primitives

• Alternatives $A = \{l, r\}$
• Decision times $t \in T = [0, \infty)$
• Data : $P \in \Delta(A \times T)$

Experiments

• Cognitive/perceptual tasks: (dots on a screen- which way are most of them going?)

• Choice tasks: (do you want an apple or a banana?)
When are decisions “more accurate?”

In cognitive tasks, **more accurate** = **more often correct**.

In choice tasks, we define **more accurate** = **more often modal** (closer to deterministic)

The correlation between decision time and accuracy depends on the decision process.

**Exogenous stopping: speed-accuracy tradeoff**
  (more accurate decisions when given more time.)
  (e.g. Liu [2013], Natenzon [2013])

**Opposite correlation seen in many experiments:** slow decisions are less likely to be correct/modal.
Data from Krajbich, Armel and Rangel (2010)

(Menus clustered by reported ranking of items)
Our explanation: a version of the “drift diffusion model” (DDM) derived from an optimal stopping problem.

- $p^i(t)$: probability of choosing $i$ conditional on stop at $t$
- $m$: modal choice

Speed-accuracy tradeoff: $p^m(t)$ increasing
(better choices if wait longer)

--------------------- independence: $p^m(t)$ constant

--------------------- complementarity: $p^m(t)$ decreasing
What generates the choice data?

**Hitting time models**

- Stochastic process $Z_t$ starts at 0.
- "boundary" $b(t)$
- Hitting time $\tau = \inf\{t : |Z(t)| \geq b(t)\}$
- Pick $l$ if $Z_\tau > 0$, $r$ if $<0$

**Fact:** any $P \in \Delta(A \times T)$ can be generated by some hitting time model- some stochastic process $Z_t$ and a constant boundary $b = 1$. 
“DDM” (“drift-diffusion model”):

\[ Z_t = \delta t + \alpha B_t \] is a diffusion with constant drift and volatility.

Now it matters if \( b \) is constant or varies with time.

**Theorem:** Suppose that \( P \) is generated by a DDM.

Then \( P \) displays a speed-accuracy tradeoff (complementarity, independence) if and only if \( b \) is increasing (decreasing, constant) in \( t \).

**Intuition:** path symmetries. In discrete time/binomial what matters is the ratio of “upjumps” (with prob \( p \)) to “downjumps” (prob. \( 1-p \)); modal choice has probability \( (p / (1 - p))^{b(t)} \).
Deriving DDMs from optimal stopping

• Unknown state $\theta = (\theta^a, \theta^b)$

• Signals $Z^i_t = \theta^i t + \alpha B^i_t$

  Interpretations: recognition of the objects, retrieving memories, creating rationalizations

• stopping time $\tau$ solves $\sup_\tau [\max\{E_\tau \theta^a, E_\tau \theta^b\} - c\tau]$  

• Assume constant flow cost for parsimony; any boundary can be rationalized if cost function unrestricted. (Theorem 6 in paper)

• Solution depends on the prior.
Simple DDM

- two states, either $\theta = (1, -1)$ or $\theta = (-1, 1)$.

- payoff +1 if action matches state else -1.

- Agent knows the utility difference between the actions but not which is right.

- posterior depends on $Z_t = Z^a_t - Z^b_t$ but not on $t$, so belief updating doesn’t slow down as data accumulates.
**Theorem** (Wald, Arrow, Blackwell and Girshik, Shirayev):
In the simple DDM \( \exists K \) s.t. \( \tau^* = \inf \{ t \geq 0 : |Z_t| \geq B \} \).

Constant boundary: so speed-accuracy neutral.

Simple DDM used to study perception tasks since at least the 70’s, on both humans and animals.

More recently used to study choice tasks (e.g. Camerer, Rangel).

**Problems with simple DDM:**

- Data shows speed-accuracy complementarity, not neutrality.
- Predicts overly long decision times for close items.
Previous ways to generate speed-accuracy complementarity:

- ad-hoc functional forms for the boundary.
- Time-varying costs (Drugovitsch et al).
- Endogenous time variation in signal intensity (Woodford)

Our project: derives a decreasing boundary from a decision problem with a prior that seems more natural than the binomial for choice tasks.
Uncertain Differences Model

- Gaussian prior $\theta^i \sim N(X_0^i, \sigma_0^2)$.

- Posterior means are $X_t^i = \frac{X_0^i + \alpha^{-2} \sigma_0^2 Z_t^i}{1 + \alpha^{-2} \sigma_0^2 t}$.

- Sufficient statistic: $(t, X_t^l - X_t^r)$.

- Updates slow as data accumulates.

When $t$ is large and posterior means are close, two reasons to stop that aren’t present in the binomial model:

- Don’t expect to learn much more (with 2 states updating doesn’t slow down).

- The two decisions are probably about as good, not much to learn even with a perfect signal. (with 2 states, believe the two payoffs are very different even if signals close.)
Theorem

(i) There is a strictly decreasing, Lipschitz continuous, strictly positive function $k(t)$ s.t. $\tau^* = \inf\{t \geq 0 : |X_t^a - X_t^b| \geq k(t)\}$.

- Follows from the principle of optimality for continuous time processes and shift invariance of the value function, which comes from the normal beliefs.

- Lipschitz continuity of boundary lets us dispense with viscosity solutions and have an exact solution to the PDE that characterizes the optimal stopping rule.
(ii) Rescaling arguments relate the boundary for any cost $c$ and signal volatility $\sigma$ to that for $c = \sigma = 1$.

- One of these uses a “more information can’t hurt” argument and yields a weak inequality, the others are equalities.

- Use these relations to show the average process

$$\int P(\delta, a, b)d\mu(\delta)$$

has speed-accuracy complementarity.

This implies the analyst will observe complementarity when the values are drawn according to the agent’s prior.

- Setting the one inequality to an equality gives a function $k$ that is “close” to the solution- both in terms of asymptotics and numerically.
Red line: predictions of simple DDM
Blue line: uncertain difference
RECAP

• Observables: Joint distribution over choices and decision times

• General DDM (Brownian signals and an arbitrary boundary): characterize when earlier decisions better.

• Uncertain-difference DDM, with Gaussian priors and optimal stopping:
  - allows agent to learn the choice is a toss-up
  - resulting boundary better fits the data than the constant boundary of simple DDM