Customer Relationship and Sales

Shouyong Shi*
Pennsylvania State University
(sus67@psu.edu)
This version: October 2014

Abstract

In a large market where sellers post the terms of trade to direct search, I prove that there exist equilibria in which buyers make repeat purchases and sellers give priority to repeat trades. Thus, customer relationships arise naturally in an equilibrium. When the buyer-seller ratio is low, the equilibrium features separation between related and unrelated individuals. When the buyer-seller ratio is high, the equilibrium is partially mixing because a newly related seller attracts both the related buyer and unrelated buyers. Customer relationships improve welfare by reducing coordination frictions and facilitating a seller to learn about the related buyer’s utility. However, an equilibrium is constrained efficient only when it is separating and when the buyer-seller ratio is low. Moreover, customer relationships lead to a theory of sales and induce price dynamics at the micro level even when market conditions are constant. I examine how market conditions affect markups, the frequency and the duration of a sale.

Keywords: Customer relationship; Sales; Directed search; Learning.

* Address: 502 Kern Building, Department of Economics, Pennsylvania State University, University Park, PA 16802, USA. Previous versions of this paper have been presented at the Search and Matching conference at the Philadelphia Federal Reserve Bank (2014), U. of Washington (2013), Penn State (2013), the Quantitative Marketing and Economics conference (Chicago Booth, 2013), the Chicago Federal Reserve Bank (2013), the Society for Economic Dynamics meeting (Seoul, 2013), Shanghai Macro Workshop (SUFE, 2013), the NBER Summer Institute (2012), Boston University (2012), the Tokyo Conference of the OEIO (2012), and the Econometric Society Winter meeting (Chicago, 2012). I thank Leena Rudanko and Pinar Yildirim for insightful discussions of the paper, and Oleksiy Kryvtsov for the conversations on the topic. Financial support from the Canada Research Chair, the Bank of Canada Fellowship, and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. The opinion expressed here is my own and does not reflect the view of the Bank of Canada.
1. Introduction

Customer relationships and repeat purchases are prevalent. In a broad survey of US firms, Blinder et al. (1998) find that the mean share of selling to repeat customers is 85%. A significant fraction of customer relationships are informal because neither the buyer nor the seller is obligated by a formal contract to continue transactions. Instead, a buyer is attracted to a seller for repeat purchases by the price and non-price instruments that the seller offers. An important non-price instrument is service priority to loyal customers. For example, a restaurant may give the reservation priority to patrons, and an airline company may allow frequent flyers to select seats in advance and board earlier. Such non-price instruments have rarely been modeled formally in economics, since the attention has been given to prices including price discrimination. In this paper, I construct a formal model to demonstrate that customer relationships with priority arise in an equilibrium in a large market with search frictions and that such relationships improve social welfare. I also analyze the implications of customer relationships on micro price dynamics and sales.

There are positive and normative reasons for constructing a formal model of customer relationships in a large market. One positive question is why customer relationships and repeat purchases are so prevalent. A satisfactory answer to this question requires a model to generate customer relationships endogenously rather than hardwire customer loyalty into buyers’ preferences.¹ A theory of endogenous customer relationships is also suitable for answering the normative question whether customer relationships improve market efficiency. Since the answer to this normative question is likely to depend on the frictions in the market, I will emphasize search frictions and information frictions. In addition, endogenous customer relationships naturally lead to a theory of sales, because a seller may use sales to attract buyers to form relationships. Thus, customer relationships may help explain the salient features of price dynamics in the micro data. In particular, prices exhibit large variability over time for the same item at a given store, sales account for a

¹Although the marketing literature has emphasized customer loyalty, it has taken customer loyalty as a primitive of the model (e.g., Blattberg and Sen, 1974).
significant part of this variability, and prices excluding sales are sticky.\footnote{See Klenow and Kryvtsov (2008) and Nakamura and Steinsson (2008). Paciello et al. (2014) find that the interaction between the customer base and search frictions is important for accounting for the price elasticity. Menzio (2008) also proposes search frictions as an explanation for sticky prices.}

The market in the model has a large number of buyers and sellers. In each period, a seller can produce one indivisible unit of a good at a cost that is common to all sellers and publicly known. Sellers direct search by posting the terms of trade and then buyers choose which seller to visit. Match failures can occur because of the lack of coordination. In a new match, the buyer’s utility is realized to be either high or low. This realization is match specific and the buyer’s private information. After a buyer and a seller trade, the two are related. A seller chooses whether to give priority to the related buyer. A relationship breaks if the related ones fail to trade with each other in a period, either endogenously as they choose to trade with someone else or exogenously as the buyer is forced by a shock to exit the market. An exiting buyer is replaced by a new unrelated buyer.

In all equilibria, a seller gives full priority to the related buyer, and a buyer visits the related seller only when the utility is high. Thus, a seller who has traded with the same buyer for two or more consecutive periods knows that the buyer has high utility. Such a seller trades exclusively with the related buyer. In contrast, a newly related seller may attract unrelated buyers in addition to the related buyer. If such a newly related seller attracts unrelated buyers, the equilibrium is partially mixing; otherwise, the equilibrium is separating. The separating equilibrium exists when the buyer-seller ratio \( b \) is low, and the partially mixing equilibrium exists when \( b \) is high.

Customer relationships improve market efficiency. By enabling a seller to give priority to the related buyer, customer relationships reduce the coordination friction. However, customer relationships are not merely a coordination device. They also facilitate a seller to learn about the related buyer’s utility and ensure ex post efficiency of trade. A seller would like to post a relatively high price to learn more about the related buyer’s utility, but a high price may discourage the buyer’s return to the seller. By offering full priority, the seller can entice the related buyer to return and, at the same time, post a high price.
In all equilibria, only the high-utility buyer returns to the seller, and so the trading priority ensures the matches with high joint surpluses to be continued.

I show that the social planner under the same constraints of search and information frictions as the market gives full priority to repeat purchases. Also as in the equilibria, the constrained social optimum features separation when the buyer-seller ratio is below a cutoff, and partially mixing otherwise. However, this cutoff is lower than in the equilibria. When the social optimum features separation, the equilibrium is also separating and socially efficient. When the social optimum features partial mixing, the separating equilibrium is socially inefficient. The partially mixing equilibrium is also inefficient because a related seller attracts an excessive number of unrelated visitors.

The separating equilibrium has the following features of prices and sales explained in section 5. First, there are price dynamics at the micro level even when the aggregate conditions are constant. An unrelated seller holds a sale to attract buyers to form a relationship, with the intention to revert to a higher regular price once he is related. A sale can last for multiple periods. Second, when the buyer-seller ratio $b$ is low, the sale price implies a negative markup. This markdown can increase in the intensive and extensive margin of demand. Third, in contrast to markups, the duration of a sale decreases monotonically when $b$ increases. In the partially mixing equilibrium, a newly related seller posts even a lower price than an unrelated seller does, but a related seller will eventually increase price after he starts to trade exclusively with the related buyer.

In both equilibria, prices will eventually be higher in repeat purchases because a seller must be compensated for the price cut used in relationship building. This natural implication of a relationship may explain why a restaurant one likes and visits often may increase price or reduce the service quality once it has obtained enough regular customers.

Three assumptions need to be motivated up-front. One is the capacity constraint on each seller that induces the trade-off between price and the trading probability. The capacity constraint is common in the service sector, e.g., restaurants and airlines. For consumer products, inventory costs constrain the number of units available in a store.
and create possible stock-out. For tractability, this capacity is set to one as in a typical search model (see section 6 for further discussion). The second assumption is that a seller must post one price for all buyers. One justification is that price discrimination may be too costly to be optimal sometimes. For example, a restaurant that wants to price discriminate may have to incur the costs of printing different menus and defending such discrimination. Another justification is that sales in the data are defined as price cuts available to all customers. By deliberately excluding price discrimination, the model can help understand how such sales are motivated by the natural notion of customer relationships as the trading priority arising from past trades.\(^3\) The third assumption is that sellers post prices and priority to direct search. Although priority posting is not common in practice, adding it to a seller’s choice simplifies the language of the model. Even if such priority is not posted, it is the equilibrium belief of all players that a related seller will give priority to the related buyer, because such priority is optimal for the seller ex post. In richer environments, even the assumption of price posting can be replaced by much weaker assumptions of commitment together with cheap talk to direct search, as shown by Menzio (2007) and Kim and Kircher (2013).

Customer relationships in this paper contrast with the locked-in effect generated by fixed costs of switching sellers (see Klemperer, 1995). Although search frictions make it difficult to find a new partner, they have important differences from switching costs. First, welfare implications are strikingly different. When switching costs make buyers locked in, they reduce efficiency and social welfare. With search frictions, repeat purchases improve efficiency and social welfare by reducing the coordination friction and enhancing ex post efficiency of trade. Second, the trading priority is not necessary for the locked-in effect with switching costs, but it is crucial for inducing repeat purchases with search frictions. Third, in switching-cost models, each seller must have a non-negligible market share in order to create the locked-in effect. With search frictions, customer relationships arise even when there are infinitely many buyers and sellers.

\(^3\)In labor market models, Lang et al. (2005) incorporate priority, while Shi (2002, 2006) and Shimer (2005) allow for priority and price discrimination. In these papers, priority is based on exogenous characteristics rather than endogenous trading histories. Also, they do not examine the dynamics of relationships.
This paper belongs in a large literature on directed search, e.g., Peters (1984, 1991), Montgomery (1991), Moen (1997), Acemoglu and Shimer (1999), Julien et al. (2000), and Burdett et al. (2001).\textsuperscript{4} To this literature, the current paper contributes by focusing on customer relationships. As an endogenous state variable, customer relationships induce price dynamics and sales that have been ignored in this literature. In addition, endogenous relational types induce price differentials among ex ante identical sellers. Moreover, price and the selling probability are \textit{positively} related across sellers in the equilibrium, even though each seller makes the trade-off between price and the selling probability.\textsuperscript{5}

This paper is also related to three strands of the literature on sales. In the first strand, each seller faces a discontinuous demand curve, and so the equilibrium has a mixed strategy in posted prices (Shilony, 1977). Some authors interpret price reductions in the mixed strategy as sales, e.g., Salop and Stiglitz (1982) and Varian (1980).\textsuperscript{6} The second strand is based on the seminal work of Sobel (1984). In this strand, a constant flow of buyers enter the market who are either high-utility buyers local to particular sellers or low-utility shoppers patiently waiting for sales. As the stock of patient shoppers reaches a critical level, some sellers cut prices to clear this stock, resulting in a one-period sale.\textsuperscript{7} The third strand contains signaling models in which a seller uses promotional sales to signal either the quality (e.g., Milgrom and Roberts, 1986) or the cost (e.g., Bagwell, 1987). These theories of sales are complementary to the one here. As additional contrasts, there are a large number of buyers and sellers in this paper, the duration of a sale is endogenous and can be longer than one period, and a sale can imply a markdown as in the data (e.g., Dutta et al., 2002). To focus on customer relationships, I deliberately abstract from the

\textsuperscript{4}Other examples are Albrecht et al. (2006), Shi (2009), Galenianos and Kircher (2009), Kircher (2009), and Gonzalez and Shi (2010), to list a few. As in the current paper, the literature of directed search has also incorporated private information, e.g., Peters and Severinov (1997), Menzio (2007), Guerrieri (2008), Guerrieri et al. (2010), Delacroix and Shi (2013), Albrecht et al. (2014) and Kim and Kircher (2013).

\textsuperscript{5}In a directed search model, Gourio and Rudanko (2011) treat the customer base as an intangible capital of a firm and examine its role in firm growth and business cycles.

\textsuperscript{6}Note that the mixed strategy can be purified when there are a large number of sellers and buyers, as in Burdett and Judd (1983).

\textsuperscript{7}Albrecht et al. (2013) extend Sobel’s (1984) model to incorporate non-stationary equilibria. Related to but different from Sobel’s (1984) model, Lazear (1986) assumes that a seller does not know the value of the good to the buyers. After failing to sell the good, a seller updates his beliefs on buyers’ valuation downward and reduces the price to clear the inventory of goods.
durability of a good and the heterogeneity or private information in sellers’ costs.

2. Directed Search with Customer Relationships

2.1. Environment of the model

Time is discrete and lasts forever. There are $N$ sellers and $bN$ buyers, where $N$ is a large number that I will take to the limit $\infty$. The buyer-seller ratio is fixed at $b \in (0, \infty)$. All individuals discount future at a rate $r > 0$. In each period, a seller can produce one indivisible unit of a good at cost $c \geq 0$. Goods are perishable, and so a seller produces only after matching with a buyer. A good is an experience good. In a new match, the buyer must consume the good in order to know the utility of the good, $u$, where $u = u_H (> c)$ with probability $\lambda \in (0, 1)$, and $u = u_L$ with probability $1 - \lambda$.\(^8\) Denote $u_e = \lambda u_H + (1 - \lambda)u_L$. The draws of $u$ are independently and identically distributed among new matches. As a match specific characteristic, the utility stays the same as long as the buyer remains matched with the same seller. Once a match breaks up, the utility in any future match will be a new draw from the distribution. The realization of $u$ is the buyer’s private information, although a seller can learn about $u$ through repeat trades.\(^9\)

Customer relationships are built on trading histories, and trading histories are publicly observed. Two individuals are related if they traded with each other in the previous period, and unrelated otherwise. A seller chooses the priority for the related buyer, $\pi \in [0, 1]$. This is the probability that the seller will choose to trade with the related buyer in the event of being visited by both the related buyer and some unrelated buyers. The priority is full is $\pi = 1$, partial is $\pi \in (0, 1)$, and non-existent if $\pi = 0$. As a conditional probability, the priority is not activated when the related seller is visited by only the related buyer or only unrelated buyers. A relationship is informal. A buyer is free to shop at any seller, and a seller gives the priority only when it is optimal to do so. Moreover, when two related

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\(^8\)A previous version of the paper (Shi, 2011) analyzes the model without such match-specific utility. Also, section 6 will briefly discuss a continuous distribution of $u$.

\(^9\)Various assumptions here can be understood with the example of a restaurant: The number of seats is limited, a meal is perishable, a chef cooks only after a customer has made the order, and the experience of a first-time customer at a restaurant is match specific.
individuals fail to trade with each other in a period, the trading history of the relationship disappears. If the two match again in the future, the match will be a new one. This assumption keeps the analysis tractable by reducing an individual’s history relevant for choices from an infinite sequence to only a few values.\textsuperscript{10}

In each period, sellers simultaneously post the terms of trade \((p, \pi)\), where \(p\) is the price of the good. Then buyers simultaneously choose which seller to visit. Search frictions arise from the lack of coordination. Some sellers may each receive more than one visitor while other sellers may receive no visitor. Let \(q\) be the expected number of unrelated visitors received by a seller, which is referred to as the \textit{queue length}. As is well known in the literature (e.g., Peters, 1991, and Burdett et al., 2001), when the market becomes large, a seller receives one or more unrelated visitors with probability \(m(q)\), where

\[
m(q) \equiv 1 - e^{-q}.
\]  

If a seller, related or unrelated, is visited only by buyers unrelated to him, the seller randomly selects one of the visitors to trade with and, ex ante, each visitor expects to be chosen with probability \(m(q)/q\). If a seller is related and visited by both the related buyer and some unrelated buyers, the related buyer is chosen with probability \(\pi\), and an unrelated buyer is chosen with probability \((1 - \pi) m(q) / q\). If only the related buyer visits the related seller, the buyer is chosen with probability one.

After trade, buyers consume. If the match is new, the buyer draws a utility level from the distribution. If the match is a repeat purchase, the utility remains the same as before. After consumption, a buyer is forced out of the market by an exit shock with probability \(1 - \sigma\), where \(\sigma \in (0, 1)\). The value of a buyer after exiting the market is normalized to zero. An exiting buyer is replaced by a new buyer who is not related to anyone.

The terms of trade direct search because buyers choose where to visit after observing all offers. Let me clarify three aspects of this search-directing role. First, a seller must treat

\textsuperscript{10}This assumption is mild because, in the equilibrium, a buyer does not continue to trade with the same seller only if the buyer has low utility. In the out-of-equilibrium event in which the seller is visited by a buyer who traded with the seller some time ago but not in the previous period, the seller knows that the buyer has low utility, and it is optimal for the seller not to give this buyer priority.
all unrelated buyers symmetrically and, in particular, the trading priority can depend only
on whether a buyer is related to the seller but not on a buyer’s identity. This symmetry
requirement captures search frictions in a large market. However, a seller can target the
related buyer because of past trades. Thus, customer relationships can help reduce search
frictions. Second, a seller is not allowed to post one price for the related buyer and another
price for unrelated buyers, as explained in the introduction. Third, the priority offer should
be optimal to the seller ex post in order to be enforceable; i.e., the seller should trade with
a visitor that yields the highest surplus ex post. Thus, the priority choice will be part of
the equilibrium belief of all individuals regardless of whether it is posted.

If a buyer revisits the related seller, the price reveals information about the buyer’s
utility, because the buyer is more likely to revisit only when the utility is high. However,
given the utility of consumption, a high price reduces the buyer’s incentive to return to
the seller. Thus, there is tension between setting the price to induce a repeat trade and
learning about the buyer’s utility. Offering priority can reduce this tension by enabling the
seller to attract the related buyer despite a high price.

2.2. Individuals’ types and the aggregate state

Trading histories generate endogenous types. At the end of a period, sellers are indexed
by $j \in \{0, 1, 2\}$. If $j = 0$, the seller is unrelated. Such a seller failed to trade in the current
period, or traded but the buyer exited the market. If $j = 1$, the seller is newly related;
i.e., the seller traded with an unrelated buyer in the current period who survived the exit
shock. If $j = 2$, the seller has traded with the same buyer in two or more consecutive
periods including the current period, and the buyer survived the exit shock. It will become
clear that a type 2 seller knows that the related buyer has high utility.

At the end of a period, buyers are indexed by $i \in \{0, 1, 2\}$. A type 2 buyer has traded
in two or more consecutive periods with the same seller and survived the exit shock. Type
1 buyers are the newly related buyers who received high utility, and so this group has

\[11\] If all sellers could costlessly target specific buyers and give priority even when the buyers are unrelated
to them, then all individuals on the shorter side of the market would be matched. See Peters (1991) and
Burdett et al. (2001) for further elaboration.
fewer buyers than the sellers of type \( j = 1 \). Type 0 buyers consist of the buyers who failed to trade in the current period, all the new entrants, and the newly related buyers who received low utility. I put the last subgroup in \( i = 0 \) instead of \( i = 1 \) because such buyers will not revisit their related sellers in an equilibrium. Of course, a type 1 seller does not know whether the related buyer is in group 1 or 0.

The aggregate state of the economy is the distribution of individuals over \((i, j)\). I focus on the equilibrium where this distribution is stationary. Let \( \rho_j \) be the fraction of type \( j \) sellers in the seller population, and \( \beta_i \) the ratio of the number of type \( i \) buyers to the total number of sellers. Denote \( V_i(\rho_1, \rho_2) \) as the market value of a type \( i \) buyer, i.e., the maximum expected value that such a buyer can obtain in the market. Denote \( J_j(\rho_1, \rho_2) \) as the market value of a type \( j \) seller. As \( \rho \) and \( \beta \), \( V \) and \( J \) are measured at the end of a period. I will suppress the dependence of these values on the aggregate state.

### 2.3. Decisions of type 0 buyers and sellers

I will show below that a type 0 buyer does not visit a type 2 seller and that a type 0 seller is not able to attract type 1 or type 2 buyers. A type 0 buyer visits type 0 sellers and, possibly, type 1 sellers.\(^{12}\) Let \( q_j \) be the queue length of type 0 buyers visiting a type \( j \) seller, where \( j \in \{0, 1\} \). Then, \( q_j > 0 \) if and only if a type 0 buyer’s expected value of visiting a type \( j \) seller is equal to the market value, \( V_0 \). Moreover, the queue lengths across sellers should add up to the available number of type 0 buyers; that is,

\[
\rho_0 q_0 + \rho_1 q_1 = \beta_0.
\]

Let \( p_0 \) be the price posted by a type 0 seller. A visitor to the seller expects to trade with probability \( \frac{m(q_0)}{q_0} \). With probability \( \lambda \), the trade yields high utility, the buyer will become type 1 and get the value \((u_H - p_0 + \sigma V_1)\), where \( \sigma \) is the probability of surviving the exit shock. With probability \( 1 - \lambda \), the trade yields low utility, the buyer will remain type 0 and get the value \((u_L - p_0 + \sigma V_0)\). Since not trading in the current period gives the buyer the value \( \sigma V_0 \), a type 0 buyer’s surplus of trade is \([u_e - p_0 + \lambda \sigma (V_1 - V_0)]\), where

\(^{12}\)If a buyer is type 0 because he received \( u_L \), he does not visit the type 1 seller related to him.
\[ u_e = \lambda u_H + (1 - \lambda) u_L. \] A type 0 buyer’s value function \( V_0 \) satisfies\(^\text{13}\)

\[
(1 + r) V_0 = \frac{m(q_0)}{q_0} [u_e - p_0 + \lambda \sigma (V_1 - V_0)] + \sigma V_0. \tag{2.3}
\]

This implies the following trade-off between the price \( p_0 \) and the queue length \( q_0 \):

\[
p_0 = u_e + \lambda \sigma (V_1 - V_0) - \frac{q_0}{m(q_0)} (1 + r - \sigma) V_0. \tag{2.4}
\]

By changing the price, a type 0 seller can induce the queue length of visitors to change according to (2.4). Thus, a seller effectively chooses \( p_0 \) and \( q_0 \), with (2.4) as a constraint.

A trade gives the seller the value, \( p_0 - c + \sigma (J_1 - J_0) \). Since a seller does not exit the market, the continuation value of a type 0 seller will be \( J_0 \). Thus, the seller’s surplus of trade is \( p_0 - c + \sigma (J_1 - J_0) \). The seller’s value function \( J_0 \) obeys:

\[
(1 + r) J_0 = \max_{(p_0, q_0)} \{ m(q_0) [p_0 - c + \sigma (J_1 - J_0)] \} + J_0, \tag{2.5}
\]

subject to (2.4). The seller takes as given the continuation values, i.e., the value functions on the right-hand sides of (2.4) and (2.5). By (2.3), (2.5) and the first-order condition of \( q_0 \), a type 0 individual’s expected surplus is

\begin{align*}
\text{for a type 0 buyer:} & \quad (1 + r - \sigma) V_0 = m'(q_0) \Delta_0 \\
\text{for a type 0 seller:} & \quad r J_0 = [m(q_0) - q_0 m'(q_0)] \Delta_0, \tag{2.6}
\end{align*}

where \( \Delta_0 \) is the joint surplus of the trade defined by

\[
\Delta_0 \equiv u_e - c + \lambda \sigma (V_1 - V_0) + \sigma (J_1 - J_0). \tag{2.7}
\]

### 2.4. Decisions of type 2 buyers and sellers

Consider a type 2 buyer and the related seller. Label the particular seller, \( \ell \), who offers \((p_2, \pi_2)\). Past trades have already revealed that a type 2 buyer has high utility of the good in the match. If the seller trades with the related buyer, the seller will remain type 2 and get the surplus \([p_2 - c + \sigma (J_2 - J_0)]\). If the seller trades with a buyer unrelated to him, the seller will become type 1 and get the surplus \([p_2 - c + \sigma (J_1 - J_0)]\). Thus, it is optimal for the seller to give full priority to the related buyer, \( \pi_2 = 1 \), if and only if

\[
J_2 > J_1. \tag{2.8}
\]

\(^{13}\)Since \( V_0 \) is measured at the end of a period, all terms on the right-hand side are discounted by \((1 + r)\).
I will prove that this condition is satisfied. A type 2 seller’s value function satisfies

\[(1 + r)J_2 = p_2 - c + \sigma (J_2 - J_0) + J_0.\] (2.9)

A trade with seller \(\ell\) gives the type 2 buyer the surplus \([u_H - p_2 + \sigma (V_2 - V_0)]\). The value function \(V_2\) satisfies: \((1 + r) V_2 = u_H - p_2 + \sigma V_2\). As the alternative to visiting seller \(\ell\), a type 2 buyer can visit an unrelated seller, which yields the value \(V_0\). Thus, the buyer will visit seller \(\ell\) with certainty if and only if \(V_2 > V_0\). It is optimal for seller \(\ell\) to set \(p_2\) to keep \(V_2\) above \(V_0\) by an arbitrarily small amount \(\varepsilon > 0\). In the limit \(\varepsilon \to 0\),

\[V_2 = V_0, \quad p_2 = u_H - (1 + r - \sigma) V_0.\] (2.10)

With this price and \(\pi_2 = 1\), a type 2 seller attracts only the buyer related to him.

2.5. Decisions of type 1 sellers and their related buyers

Consider a type 1 seller \(\ell\) who offers \((p_1, \pi_1)\). This seller does not know whether the related buyer received \(u_H\) or \(u_L\). If the related buyer received \(u_L\), trading with seller \(\ell\) yields the value \((u_L - p_1 + \sigma V_0)\). To rule out the uninteresting case, I assume that \(u_L\) is so low that the related buyer with \(u_L\) does not visit the seller related to him even if the seller gives full priority to the buyer. That is,

\[p_1 > u_L - (1 + r - \sigma) V_0.\] (2.11)

Under this assumption, a repeat purchase reveals that the related buyer has high utility.

Seller \(\ell\) may or may not be visited by unrelated buyers. If seller \(\ell\) is visited by unrelated buyers and trades with one of them, the seller will remain type 1 and get the surplus \([p_1 - c + \sigma (J_1 - J_0)\]]. If seller \(\ell\) trades with the related buyer, the seller will become type 2 and get the surplus \([p_1 - c + \sigma (J_2 - J_0)\]]. Since (2.8) is satisfied, trading with the related buyer yields a higher surplus to seller \(\ell\) than with an unrelated buyer, and so \(\pi_1 = 1\).

A trade with seller \(\ell\) gives the related type 1 buyer the surplus \([u_H - p_1 + \sigma (V_2 - V_0)]\). Because \(V_2 = V_0\), a type 1 buyer’s value function \(V_1\) satisfies:

\[(1 + r) V_1 = u_H - p_1 + \sigma V_0.\] (2.12)
For a type 1 buyer to visit seller \( \ell \) instead of a type 0 seller, it is necessary and sufficient that \( V_1 > V_0 \). Including the limit where \( V_1 = V_0 \), I rewrite this requirement as

\[
p_1 \leq u_H - (1 + r - \sigma) V_0 \quad (= p_2).
\] (2.13)

A type 0 visitor to a type 1 seller expects to trade if and only if the seller’s related buyer received \( u_e \) and no other type 0 visitor is chosen by the seller. This trading probability is \( (1 - \lambda) \frac{m(q_1)}{q_1} \), where \( q_1 \) is the queue length of type 0 buyers for the seller. Similar to a trade with a type 0 seller, a trade with a type 1 seller gives a type 0 buyer the surplus \( [u_e - p_1 + \lambda \sigma (V_1 - V_0)] \). Because \( q_1 > 0 \) if and only if the buyer is indifferent between visiting a type 1 and a type 0 seller, then

\[
(1 - \lambda) \frac{m(q_1)}{q_1} [u_e - p_1 + \lambda \sigma (V_1 - V_0)] = (1 + r - \sigma) V_0 \quad \text{if } q_1 > 0.
\] (2.14)

A type 1 seller succeeds in a trade with the related buyer with probability \( \lambda \), and with an unrelated buyer with probability \( (1 - \lambda) m(q_1) \). Given the surplus of each trade calculated above, a type 1 seller’s value function \( J_1 \) obeys:

\[
(1 + r) J_1 - J_0 = \max_{(p_1, p_2)} \{ \lambda [p_1 - c + \sigma (J_2 - J_0)]
+ (1 - \lambda) m(q_1) [p_1 - c + \sigma (J_1 - J_0)] \}
\] (2.15)

subject to (2.14) and (2.13). Again, the seller takes the continuation values as given.

3. Equilibrium with Customer Relationship

3.1. Equilibrium definition

The distribution of individuals is endogenous. Use the subscript +1 to indicate next period. With \( \rho_0 = 1 - \rho_1 - \rho_2 \), the distribution obeys:

\[
\begin{align*}
\rho_{1, +1} - \rho_1 &= \rho_0 \sigma m(q_0) - \rho_1 [1 - \sigma (1 - \lambda) m(q_1)], \\
\rho_{2, +1} - \rho_2 &= \rho_1 \lambda \sigma - \rho_2 (1 - \sigma).
\end{align*}
\] (3.1)

The flow into type 1 sellers is \( \rho_0 \sigma m(q_0) \), since a type 0 seller becomes type 1 after the seller trades with a buyer who survives the exit shock. A type 1 seller remains type 1 after a period only when the seller succeeds in a trade with a type 0 buyer who survives the exit shock. Since the probability of this event is \( \sigma (1 - \lambda) m(q_1) \), the outflow from type 1 sellers
is $\rho_1 [1 - \sigma (1 - \lambda) m(q_1)]$. For type 2 sellers, the inflow is type 1 sellers who succeed in trading with their related buyers that survive the exit shock, as given by $\rho_1 \lambda \sigma$, and the outflow is the sellers whose related buyers exit the market, as given by $\rho_2 (1 - \sigma)$.

The distribution of buyers, represented by $(\beta_0, \beta_1, \beta_2)$, is given as follows:

$$\beta_2 = \rho_2, \quad \beta_1 = \rho_1 \lambda, \quad \beta_0 = b - \beta_1 - \beta_2.$$  \hspace{1cm} (3.2)

The number of type 2 buyers is equal to the number of type 2 sellers. Only a fraction $\lambda$ of the newly related buyers are type 1. The overall buyer-seller ratio is $b$.

An equilibrium with customer relationships consists of buyers’ value functions $V_i$ for $i \in \{0, 1, 2\}$, sellers’ choices $(p_j, \pi_j)$ and value functions $J_j$ for $j \in \{0, 1, 2\}$, queue lengths $(q_0, q_1)$, and the distribution of individuals $(\rho_0, \rho_1, \rho_2, \beta_0, \beta_1, \beta_2)$ that satisfy the following requirements:\footnote{Under the assumption that all buyers of the same type make the same decision, including the response to a seller’s deviation, buyers’ decisions matter for the equilibrium only through the queue lengths.} (i) $(p_0, q_0, V_0, J_0)$ satisfy (2.2) - (2.6), $(p_2, V_2, J_2)$ satisfy (2.9) - (2.10), and $(p_1, q_1, V_1, J_1)$ satisfy (2.12) - (2.15); (ii) $J_2 > J_1$, and so $\pi_1 = \pi_2 = 1$; (iii) $(\rho_0, \rho_1, \rho_2)$ and $(\beta_0, \beta_1, \beta_2)$ are time invariant and satisfy (3.1), (3.2) and $\rho_0 = 1 - \rho_1 - \rho_2$.

The equilibrium is separating (SE) if $q_1 = 0$, and partially mixing (PME) if $q_1 > 0$. The mixing is partial because high-utility buyers visit only their related sellers. In both the SE and PME, it is optimal for a related seller to give full priority to the related buyer.

### 3.2. Existence of an equilibrium

In the PME, $q_1 > 0$, and (2.14) solves $p_1$ as a function of $q_1$. Substituting this function

and (2.6), I express a type 1 seller’s maximization problem as

$$\begin{align*}
(1 + r)J_1 - J_0 &= \max_{q_1 \geq 0} [\lambda \sigma (J_2 - J_1) + A(q_1, q_0) \Delta_0],
\end{align*}$$  \hspace{1cm} (3.3)

where $\Delta_0$ is defined in (2.7) and

$$A(q_1, q_0) \equiv [\lambda + (1 - \lambda) m(q_1)] \left(1 - \frac{q_1 m'(q_0)}{(1 - \lambda) m(q_1)}\right).$$  \hspace{1cm} (3.4)
If \( q_1 > 0 \), then \( q_1 \) satisfies the first-order condition, \( q_1 = h^{-1}(q_0) \), where

\[
h(q) \equiv q + \ln \left\{ \frac{1}{1 - \lambda} \left[ 1 + \frac{\lambda [m(q) - q m'(q)]}{(1 - \lambda) m^2(q)} \right] \right\}. \tag{3.5}
\]

\( h \) is strictly increasing for all \( q \geq 0 \), with \( h'(q) > 0 \). As an alternative to the choice \( q_1 = h^{-1}(q_0) \), a type 1 seller can set \( p_1 = p_2 \) to induce \( q_1 = 0 \) and obtain the expected surplus \( \lambda [p_2 - c + \sigma (J_2 - J_0)] \). Partial mixing is optimal for a type 1 seller if this expected surplus is lower than that in (3.3), that is, if

\[
A(q_1, q_0) \Delta_0 > \lambda [p_2 - c + \sigma (J_1 - J_0)]. \tag{3.6}
\]

Note that this requirement cannot be replicated by requiring a type 1 seller’s payoff in (3.3) to be higher at \( q_1 = h^{-1}(q_0) \) than at \( q_1 = 0 \), because the seller’s payoff is not continuous at \( q_1 = 0 \). At \( q_1 = 0 \), the price is no longer constrained by (2.14), and the seller’s payoff has a discrete jump as \( p_1 \) increases to \( p_2 \).

In the SE, \( q_1 = 0 \), and a type 1 seller sets \( p_1 = p_2 \). Then, \( V_1 = V_0 \), and \( J_1 \) satisfies:

\[
(1 + r)J_1 - J_0 = \lambda [p_2 - c + \sigma (J_2 - J_0)]. \tag{3.7}
\]

For the SE, it should not be profitable for the seller to deviate from \( q_1 = 0 \) to \( \tilde{q}_1 = h^{-1}(q_0) \). That is, (3.6) should be violated with \( q_1 \) being replaced by \( \tilde{q}_1 \).

Condition (3.6) depends on \( q_0 \) and \((J_1, J_0, V_1, V_0)\) that, in turn, depend on whether the equilibrium is PME or SE. This condition must be checked separately for each equilibrium because a type 1 seller’s payoff function is discontinuous at \( q_1 = 0 \). Appendix A determines the value functions in the two equilibria. For \( q_0 \), I use (2.2), (3.2) and the steady state version of (3.1) to derive:

\[
b = R(q_0, q_1) \equiv \frac{\lambda}{1 - \sigma} q_1 + \left[ 1 - \sigma (1 - \lambda) m(q_1) \right] \frac{q_0}{\sigma m(q_0)}. \tag{3.8}
\]

Let \( q_A \) solve \( b = R(q_A, 0) \) and \( q_B \) solve \( b = R(q_B, h^{-1}(q_B)) \). Then, \( q_0 = q_A \) in the SE, and \( q_0 = q_B \) in the PME. The following proposition is proven in Appendix A:
Proposition 3.1. Define $B_a = R(q_a, 0)$ and $B_b = R(q_b, h^{-1}(q_b))$, where $q_a$ and $q_b$ are defined in Lemma A.2. Assume $u_L = c$ for simplicity. (i) $q_A$ and $q_B$ exist, are unique, and increase in $b$. Moreover, $q_A \geq q_B$. (ii) The SE exists if and only if $b \leq B_a$, while the PME exists if and only if $b > B_b$. (iii) $J_2 > J_1$ in both equilibria. (iv) $p_0 < p_1 = p_2$ in the SE, and $p_1 < p_0 < p_2$ in the PME.

Because $J_2 > J_1$, it is optimal for a related seller to give full priority to the related buyer. A type 2 seller obtains a higher value than a type 1 seller in an equilibrium because all the choices available to a type 1 seller are also available to a type 2 seller. In addition, a type 2 seller knows that the related buyer has high utility, but a type 1 seller only knows that the related buyer has high utility with probability $\lambda$.

The existence conditions of the two equilibria are explained as follows. Because priority is common knowledge in the equilibrium, an unrelated buyer is reluctant to visit a seller who gives priority to the related buyer. Thus, giving priority to the related buyer reduces a type 1 seller’s ability to compete for unrelated buyers. To attract an unrelated buyer, a type 1 seller must set the price sufficiently low to compensate the buyer for the low trading probability. This downward pressure on the price increases as the buyer-seller ratio $b$ decreases. If $b < B_a$, the price set by a type 1 seller to attract unrelated buyers must be so low that it is not optimal. Instead, such a seller can gain by setting the price to be high to attract only the related type 1 buyer. This generates the SE. If $b > B_b$, competition for unrelated buyers is weak so that a type 1 seller can attract both the related buyer and unrelated buyers. This generates the PME.

It is ambiguous whether $B_a$ is higher or lower than $B_b$. When $B_b < B_a$, the SE and PME coexist for $b \in (B_b, B_a)$. This coexistence arises from two opposite effects on a type 1 seller from other type 1 sellers’ competition for unrelated buyers. One is the direct effect. When other type 1 sellers attract unrelated buyers, the intensified competition reduces an individual type 1 seller’s incentive to attract unrelated buyers. The other is a spillover

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15 When the two equilibria coexist, the welfare ranking between the two is ambiguous analytically. Also, if $B_b > B_a$ and $b \in (B_a, B_b)$, there is no equilibrium in which all individuals of the same type use the same strategy. However, the following asymmetric equilibrium can exist: a fraction of type 1 sellers use a low price to attract $q_1 > 0$ and the remaining fraction of type 1 sellers use a high price to induce $q_1 = 0$. 15
effect through a type 2 seller’s payoff. The intensified competition for unrelated buyers increases the value of a related buyer’s outside option and, thereby, reduces the price \( p_2 \). As the gain to becoming a type 2 seller falls, the incentive increases for a type 1 seller to remain type 1 by attracting unrelated buyers. When this spillover effect is strong, the two equilibria coexist. In this case, if other type 1 sellers attract unrelated buyers, so does an individual type 1 seller and, if other type 1 sellers attract only their related buyer, so does an individual type 1 seller. The following example illustrates the coexistence:

**Example 1.** \( u_H = 1, u_L = c = 0.35, \sigma = 0.95 \) and \( r = 0.001 \). With these parameter values, Figure 1 plots the two cutoffs \( B_a \) and \( B_b \) as functions of \( \lambda \), where \( \sigma 1 = 0.95 \). For \( \lambda < 0.49 \), \( B_b > B_a \), and so the two equilibria do not coexist. For \( \lambda > 0.49 \), \( B_b < B_a \), and so the two equilibria coexist in the narrow region \( b \in (B_b, B_a) \).

![Figure 1. Existence of an equilibrium](image)

Proposition 3.1 also states \( p_2 > p_0 \). Once a seller has deduced that the related buyer has high utility, the seller increases price. In the SE, this result implies \( p_1 > p_0 \), because \( p_1 = p_2 \) in the SE. In contrast, \( p_1 < p_0 \) in the PME; that is, a (related) type 1 seller posts a price even lower than an unrelated seller does. Although striking, this result arises intuitively from the feature that priority is common knowledge. As explained above, giving priority to the related buyer is optimal for a related seller, but the priority also severely ties the hands of a type 1 seller when it comes to attracting unrelated buyers. Because an unrelated buyer knows that a type 1 seller will give the related buyer priority, the unrelated buyer is willing to visit a type 1 seller only when the price is lower than the one
charged by a seller who does not have a related buyer. Thus, customer relationships do not always lead to higher prices immediately—they lead to higher prices eventually when related individuals choose to trade with each other exclusively.

4. Social Optimum

Consider a social planner who is constrained by the same search and information frictions as the market is. On search, the planner must treat all individuals of the same type symmetrically and, hence, cannot make the allocation depend on an individual’s identity beyond customer relationships. The matching probabilities are given by the same formulas as in the equilibrium. On information, the planner does not know whether a newly related buyer has high or low utility. In the market, prices reveal information about a buyer’s utility and induce the desirable outcome that a buyer with high utility returns to the related seller for repeat purchases. For the planner to induce this outcome, I assume that the planner allocates the joint surplus of a match to the buyer, subject to the buyer’s participation. If a buyer has high utility, it is incentive compatible for the buyer to return to the same seller. If a buyer has low utility, the joint surplus of a match with the related seller is zero under the assumption \( u_L = c \), and so the buyer will participate in a match with an unrelated seller instead.

As in the market, a buyer’s type is \( i \in \{0, 1, 2\} \) and a seller’s type is \( j \in \{0, 1, 2\} \). The planner knows that a type 2 buyer has high utility and allocates only this buyer to visit the related seller. To a newly related buyer, the planner gives full priority in the repeat trade. However, because the planner does not know whether the buyer has high utility \( (i = 1) \) or low utility \( (i = 0) \), the planner does not know whether the buyer will participate in the repeat match. Of type 0 buyers, the planner chooses a queue length \( q_0 \) to visit each type 0 seller and \( q_1 \) to visit each type 1 seller. An unrelated buyer visiting a type 1 seller is chosen to trade only when the seller is not visited by the related buyer, in which case each unrelated visitor is chosen with equal probability.

Let \( W(\rho_1, \rho_2) \) be the social welfare function that measures the sum of expected social
surpluses over the matches, normalized by the total number of sellers. In an arbitrary period, the number of trades involving type 1 and type 2 buyers is \(( \rho_1 \lambda + \rho_2 ) N\), and such a trade yields the joint surplus \((u_H - c)\). The trades involving type 0 buyers have two groups. One is with type 1 sellers whose related buyers have low utility of the sellers’ goods and choose to trade elsewhere. The number of trades involving such sellers is \(\rho_1 (1 - \lambda) m(q_1) N\). The other is with type 0 sellers; the number of these trades is \(\rho_0 m(q_0) N\). A trade with type 0 buyer yields the surplus \(u_e - c\). Thus, the social welfare function and the planner’s maximization problem obey:

\[
(1 + r) W(\rho_1, \rho_2) = \max_{(q_0, q_1)} \{[\rho_1 (1 - \lambda) m(q_1) + (1 - \rho_1 - \rho_2) m(q_0)](u_e - c) + (\rho_1 \lambda + \rho_2)(u_H - c) + W(\rho_1, 1, \rho_2, 1)\}
\]

subject to \(q_0 \geq 0, q_1 \geq 0, (3.1)\), and the adding-up constraint: \((1 - \rho_1 - \rho_2)q_0 + \rho_1 q_1 = b - \rho_2 - \rho_1 \lambda\). The last constraint comes from substituting \(\beta_0 = 1 - \rho_2 - \rho_1 \lambda\) into (2.2).

Define \(h_e(q) \equiv m^{-1}(m'(q)\frac{1}{1-\lambda})\) and \(B_e \equiv R(m^{-1}(1 - \lambda), 0)\), where \(R(q_0, q_1)\) is defined by (3.8). The following proposition is proven in Appendix B:

**Proposition 4.1.** The social optimum gives full priority to repeat trades. If \(b \leq B_e\), the social optimum consists of \(q_1 = 0\) and \(q_0\) that solves \(b = R(q_0, 0)\). If \(b > B_e\), the social optimum consists of \(q_1 = h^{-1}_e(q_0) > 0\) and \(q_0\) that solves \(b = R(q_0, h^{-1}_e(q_0))\). Moreover, \(h_e(q) > h(q)\) for all \(q \geq 0\), and \(B_e < \min\{B_a, B_b\}\). Thus, if \(b \leq B_e\), the equilibrium (SE) is socially efficient. If \(b > B_e\), the SE is inefficient because the social optimum features partial mixing in this case. The PME is also inefficient because it has an inefficiently high \(q_1\) and an inefficiently low \(q_0\).

The social optimum gives full priority to repeat trades, as in an equilibrium. Such priority is socially efficient for two reasons. First, the priority reduces the coordination friction. Match failures occur in the market because it is costly for sellers to each identify a distinct buyer and give the trading priority to the buyer. Customer relationships reduce match failures by enabling a seller to offer priority to the related buyer. Second, full priority induces ex post efficiency in a trade. Since a repeat buyer has high utility, a trade with the repeat buyer yields a higher joint surplus than a trade with a new customer.

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The social optimum may require separation or partial mixing. The efficient queue lengths maximize the marginal contribution of an unrelated buyer to the number of trades. An unrelated buyer increases the number of matches only when a seller is not already matched. This marginal contribution is \(1 - m(q_0) = m'(q_0)\) in the matching with type 0 sellers, and \((1 - \lambda)m'(q_1)\) in the matching with type 1 sellers. Whenever possible, the planner wants to equalize the marginal contributions of an unrelated buyer in the two types of matches. Such equalization is possible when \(1 - \lambda > m'(q_0)\), in which case the social optimum requires partial mixing. That is, \(q_1 > 0\) in this case and satisfies \((1 - \lambda)m'(q_1) = m'(q_0)\). If \(1 - \lambda \leq m'(q_0)\), the equalization is not possible, because an unrelated buyer has a higher marginal contribution to the matches with type 0 sellers than with type 1 sellers even if \(q_1 = 0\). In this case, the social optimum requires separation that allocates unrelated buyers to match only with type 0 sellers; i.e., \(q_1 = 0\). The separating condition, \(1 - \lambda \leq m'(q_0)\), is equivalent to \(b \leq B_e\). Intuitively, when the number of buyers is small, an unrelated buyer increases the number of matches with type 0 sellers by more than with type 1 sellers.

The cutoff \(B_e\) is lower than the equilibrium counterparts, \(B_a\) and \(B_b\). If \(b \leq B_e\), the equilibrium features separation, as the social optimum does. Also, the queue length is efficient, and so is the equilibrium in this case. If \(b > B_e\), the equilibrium can be the SE or PME. Both equilibria are socially inefficient in this case. The SE is inefficient because it does not feature partial mixing as the social optimum requires in this case. The PME is inefficient because it allocates an excessive number of type 0 buyers to match with type 1 sellers instead of with type 0 sellers. Recall that a type 1 seller in the PME must set the price lower than a type 0 seller does in order to attract unrelated buyers (see Proposition 3.1). This excessively low price results in an inefficiently high number of unrelated buyers being allocated to type 1 sellers and away from type 0 sellers in the equilibrium relative to the social optimum.

The inefficiency arises from the tension between the price that is optimal for a type 1 seller in a repeat trade and the price that is optimal for attracting unrelated buyers. In a repeat trade, a type 1 seller wants to set the price as high as \(p_2\) to extract the surplus.
of the trade. However, to attract unrelated buyers, a type 1 seller needs to set the price lower than $p_0$. This tension explains why a type 1 seller in the equilibrium clings to the separation with the high price $p_2$ in a wider interval of $b$ than in the social optimum, i.e., $\min\{B_a, B_b\} > B_e$. It also explains why a type 1 seller underprices to attract an inefficiently large queue length of unrelated buyers in the equilibrium once $b$ is large enough to support partial mixing. If a type 1 seller could post prices conditional on buyers’ type, then the seller would use a relatively low price to attract unrelated buyers and, at the same time, use full priority to attract the related buyer and charge the high price $p_2$ for this trade. Such price discrimination improves efficiency, as being illustrated by Shi (2002, 2006) and Shimer (2005) in related models of the labor market with directed search. However, such price discrimination may not always be possible in reality, as explained in the introduction.

5. Relationships, Sales and Markups

This section focuses on the SE, with a brief discussion on the PME at the end.

A relationship forms when a buyer trades and survives the exit shock. A relationship breaks up endogenously in the first period of the relationship when the buyer receives low utility, and exogenously in any period when the buyer exits the market. A related seller posts the high price $p_1 = p_2$ and an unrelated seller posts the low price $p_0 < p_2$. The high price $p_2$ is paid not only by buyers in most trades in any given period but also most often over time to any given seller. Thus, I call $p_2$ the regular price and $p_0$ the sale price. By holding a sale, a seller intends to attract buyers to form a relationship, but the relationship will pay off only in the future through a higher (regular) price. This intertemporal motive for holding sales generates price dynamics at the micro level even though market conditions are constant. A seller posts the regular price as long as the related buyer keeps visiting him. Once a seller loses a relationship, he holds a sale at price $p_0$, with the intention to revert to a higher price $p_2$ after he forms a relationship.

In each period, the number of trades is $N [p_0 m(q_0) + \rho_1 \lambda + \rho_2]$ at the regular price and $N p_0 m(q_0)$ at the sale price. The frequency of a sale is $p_0 m(q_0) / [p_0 m(q_0) + \rho_1 \lambda + \rho_2]$. 
Since a sale ends when the seller becomes type 1, the duration of a sale is \( \frac{1}{\sigma m(y_0)} \), which is endogenous. A sale lasts for more than one period not only because an unrelated seller trades with probability less than one, but also because an unrelated seller will continue to hold a sale after a trade if the buyer in the trade exits the market. Denote the markup implied by price \( p_j \) as \( mkp_j = \frac{p_j}{c} - 1 \) for \( j \in \{0, 1, 2\} \). The following proposition is proven in the Supplementary Appendix:

**Proposition 5.1.** The SE has the following features:

(i) When \( \lambda \) is sufficiently close to 1, a seller who posts the regular price is more likely to succeed in trade than a seller who posts the sale price.

(ii) The frequency of a sale is \( \frac{1-\sigma}{1-\sigma+\lambda\sigma} \), which decreases in \( \lambda \) and \( \sigma \).

(iii) The duration of a sale is \( \frac{1}{\sigma m(y_0)} \), which decreases in \( b \).

(iv) Assume \( u_L = c \). There exists \( B_0 > 0 \) such that \( mkp_0 < 0 \), and \( mkp_0 \) decreases in \( u_H/c \), if and only if \( b < B_0 \). If \( \lambda \leq \frac{1+r}{\sigma} - 1 \), then \( mkp_0 \) increases in \( b \). If \( \lambda > \frac{1+r}{\sigma} - 1 \), then there exists \( B_1 \in (0, B_0) \) such that \( mkp_0 \) decreases in \( b \) if and only if \( b < B_1 \).

Part (i) states that when customer relationships are important in the sense that \( \lambda \) is high, posting the sale price increases future trades for a seller by increasing the probability of establishing the customer relationship. That is, the cross-sectional relationship between price and the selling probability is positive at any given time. This positive cross-sectional relationship does not contradict the trade-off faced by a seller in directed search, which requires the two variables to be negatively related. The consistency is explained by the fact that sellers are endogenously heterogeneous in customer relationships. A related seller has a related buyer but an unrelated seller does not. By offering priority to explore the relationship, a related seller is able to both charge a higher price and sell a good more quickly than an unrelated seller does. Given a seller’s type, if the seller increases price, the selling probability does fall. Thus, to test whether the trade-off between price and the selling probability exists in a data, one must control for endogenous heterogeneity among sellers, such as the trading history that determines the seller’s type in this model.

The frequency of a sale depends on only two parameters, \( \lambda \) and \( \sigma \), and decreases in
both. A high $\lambda$ increases the likelihood of forming a relationship and, hence, reduces the need to hold a sale. A high $\sigma$ reduces the frequency of a sale by making a relationship last longer. The duration of a sale decreases in $b$, because a high buyer-seller ratio increases a type 0 seller’s trading probability and, thereby, ends a sale more quickly. Note that both the frequency and the duration of a sale are independent of $u$ and $c$.

Prices and markups do not behave in the familiar way. Markups are neither constant nor always positive. When $b < B_0$, the sale price implies a negative markup, and this markdown increases in the intensive margin of demand measured by $u_H/c$. These results are intuitive. Relationships are important when buyers are relatively scarce in the market in the sense $b < B_0$. In order to attract buyers to form a relationship in this case, a seller cuts price below the marginal cost, resulting in the markdown. For the seller, the temporary loss of the markdown will be more than compensated by the future rent (return) of a relationship that can be measured by $J_2 - J_0$. Similarly, an increase in $u_H/c$ increases the rent of a relationship to a seller and increases the incentive to cut price to form a relationship. Thus, the markdown increases in $u_H/c$ when buyers are scarce in the market. After $b$ surpasses the threshold $B_0$, the need to mark down disappears.

The markup implied by the sale price has a V-shaped dependence on the extensive margin of demand, $b$, if $\lambda$ is sufficiently large. Starting at low values of $b$, the sale price implies a markdown that first increases in $b$ and then, after $b$ surpasses the threshold $B_1$, decreases in $b$. After $b$ surpasses a higher threshold $B_0$, the sale price implies a positive markup that increases in $b$. Since an increase in $b$ makes it easier for a seller to find a match, it seems puzzling why the markdown increases in $b$ when $b$ is small. The key to resolving the puzzle is to notice that an increase in $b$ reduces the outside option of a related buyer. This increases a seller’s gain from a relationship and motivates an unrelated seller to increase the price cut in order to form a relationship. For this effect to be the dominant force on the markup, buyers must be scarce in the market and a trade must be likely to result in repeat purchases, i.e., $b$ must be small and $\lambda$ must be high. In this case, an increase in $b$ increases the markdown implied by the sale price.\textsuperscript{16} When either $b$ is high or

\textsuperscript{16}With scanner data, Chevalier et al. (2003) find that prices of particular items fall in religious holidays.
\( \lambda \) is low, an increase in \( b \) increases the markup.

A lesson here is that with search frictions, non-price components such as priority play an important role in determining how an equilibrium responds to changes in demand. Relative to prices and markups, the frequency and duration of a sale are more reliable indicators of such changes.

Now I briefly discuss the PME. With partially mixing, there are two types of sales. One is by type 0 sellers. The other is the deeper sale by type 1 sellers, since \( p_1 < p_0 \) (see Proposition 3.1). Prices, markups and the sale duration are complicated analytically. As an illustration, consider Example 1 and set \( \lambda = 0.5 \). Then the PME exists when \( b > B_b = 1.16 \). Because \( b \) is relatively large in this case, demand is strong, and so the markups implied by \((p_0, p_1, p_2)\) are all positive and increase in \( b \). Moreover, as \( b \) increases, the duration of a sale at \( p_0 \) declines and the duration of a sale at \( p_1 \) increases.

### 6. Concluding Remarks

This paper has constructed a theory of customer relationships and sales in a large market. I have shown that customer relationships with service priority improve market efficiency and social welfare. In addition, endogenous customer relationships lead to a theory of sales. The model captures the features in the data that prices exhibit a large variability over time for the same item by a given seller and sales contribute to this variability. Dynamics and endogenous entry of sellers can be incorporated; see Shi (2011) for such analyses where a buyer’s utility is public information. The extension will be useful for studying business cycles with micro price dynamics. Moreover, because an increase in demand can reduce markups in the model, markups can be counter-cyclical, a property central to business cycle models that focus on demand shocks. Furthermore, a macro model with sales should emphasize the duration of a sale, because this duration responds to market conditions more accurately than prices do.

Two other extensions are worth pursuing when taking the model to the data. One is to...
allow a buyer’s utility to be distributed continuously. In this extension, in contrast to the
model here, a buyer’s utility is never completely revealed, because a seller stops learning
after a few periods. A relationship can break up endogenously even after a repeat trade.
However, as in the model here, the extended model features a finite number of price levels
in an equilibrium. This extension will be undertaken in a sequel. Another extension is
to relax the capacity constraint so that a seller can sell multiple units. As illustrated by
Burdett et al. (2001), an increase in the capacity can increase, rather than decrease, the
coordination friction. When every seller’s capacity is increased, a smaller fraction of sellers
are needed to satisfy the market and, without coordination, all buyers may end up with
this smaller fraction of sellers. The increased competition among sellers will make customer
relationships even more important for sellers than when each seller’s capacity is one unit.
A natural conjecture is that a seller increases price only after the number of buyers related
to the seller exceeds a threshold and holds a sale only after the number of buyers related
to the seller falls below another threshold. Although the equilibrium in this extension is
unlikely to be tractable analytically, numerical analyses can be conducted.
Appendix

A. Proof of Proposition 3.1

Before proving the proposition, I state the following two lemmas. The proofs of these lemmas are omitted here but can be found in the Supplementary Appendix:

Lemma A.1. Define $h(q)$ by (3.5) and $t(q) = \frac{m(q) - m'(q)}{m(q)}$. Then, (i) $m'(h(q)) < (1 - \lambda) \frac{m(q)}{m'}$; (ii) $\frac{m(h(q))}{h(q)} > (1 - \lambda) \frac{m(q)}{m'}$; (iii) $h'(q) < (1 - \lambda) \frac{m'(q)}{m'(h(q))}$ and $t'(q) \leq t(q)$.

Lemma A.2. Denote $\hat{\sigma} = \frac{\sigma}{1+r}$ and

\[
\begin{align*}
g_a(\hat{q}_1, q_0) & = \frac{1}{m_0'} - \lambda + \hat{\sigma}(1 - \lambda) \left( \frac{m_0}{m_0'} - q_0 \right) - \frac{A(\hat{q}_1, q_0)/m_0'}{1 - \hat{\sigma}(1 - \lambda)}, \\
g_b(q_1, q_0) & = \frac{1}{m_0} - \lambda + \hat{\sigma}(1 - \lambda) \left( \frac{m_0}{m_0'} - q_0 \right) - \left[ \frac{1}{1 - \sigma(1 - \lambda)} + \frac{\hat{\sigma}(1 - \lambda)m_1}{\lambda(1 + \lambda)m_1} \right] \frac{A(q_1, q_0)}{m_0'}. \tag{A.2}
\end{align*}
\]

For $i \in \{a, b\}$, there exists a unique $q_{1i} > 0$ such that $g_i(q_{1i}, h(q_{1i})) = 0$. Define $q_i = h(q_{1i})$ for $i \in \{a, b\}$. Then, $g_i(h^{-1}(q_0), q_0) > 0$ if and only if $q_0 < q_i$. Moreover, $q_a > q_b > h(0)$.

In both equilibria, $J_2$ satisfies (2.9), $(V_2, p_2)$ satisfy (2.10), and $(V_0, J_0)$ satisfy (2.6). Use these equations and $\Delta_0$ defined in (2.7) to solve:

\[
\begin{align*}
J_2 & = \frac{p_2 - c + (1 - \sigma)\lambda_0}{1 + \sigma}, & p_2 & = u_H - m'(q_0)\Delta_0, \\
rJ_0 & = [m(q_0) - q_0m'(q_0)]\Delta_0. \tag{A.3}
\end{align*}
\]

The SE: $p_1 = p_2$, $V_1 = V_0$, and $J_1$ satisfies (3.7). Using (2.9) and (3.7), I solve $J_1$ and combine with (2.5) to get

\[
J_1 - J_0 = \frac{1}{1 - \hat{\sigma}} \left[ \lambda(p_2 - c) - [1 - \hat{\sigma}(1 - \lambda)] rJ_0 \right], \tag{A.4}
\]

where $\hat{\sigma} = \frac{\sigma}{1+r}$. Substituting this expression, (A.3), the fact $V_1 = V_0$ and the assumption $u_L = c$ into (2.7) yields:

\[
\Delta_0 = \frac{\lambda(u_H - c)/\hat{\sigma}}{\frac{1}{\hat{\sigma}} + 1 + \lambda m_0' + [1 - \hat{\sigma}(1 - \lambda)](m_0 - q_0m_0')} \tag{A.5}
\]
If a type 1 seller deviates from the separating strategy to attract unrelated buyers in addition to the related buyer, the price is \( \tilde{p}_1 = p(\tilde{q}_1) \), where \( p(q) \) is defined by (A.7) below, and the queue length of unrelated visitors to the seller is \( \tilde{q}_1 = h^{-1}(q_0) \), where \( h(q) \) is defined by (3.5). If \( q_0 \leq h(0) \), then \( \tilde{q}_1 \leq 0 \), and so the deviation does not succeed. Suppose \( q_0 > h(0) \). The deviation is not profitable if and only if

\[
A(\tilde{q}_1, q_0) \Delta_0 \leq \lambda [p_2 - c + \sigma (J_1 - J_0)]. \tag{A.6}
\]

Substituting \( u_L = c \), as assumed in the current proposition, and substituting \((J_1 - J_0)\) and \(\Delta_0\), I rewrite (A.6) as \( g_a(\tilde{q}_1, q_0) \geq 0 \), where \( g_a \) is defined in (A.1). By Lemma A.2, there exists a unique \( q_a > h(0) \) such that (A.6) is satisfied if and only if \( q_0 \leq q_a \).

In the SE, \( q_1 = 0 \), and \( q_0 \) solves \( b = R(q_0, 0) \), where \( R \) is defined in (3.8). Because \( m(q) \) and \( \frac{q}{m(q)} \) increase in \( q \), it is easy to verify that \( R(q_0, 0) \) is strictly increasing in \( q_0 \). Moreover, using \( m(0) = 0 \), \( \lim_{q \to 0} \frac{m(q)}{q} = 1 \) and \( m(\infty) = 1 \), I can verify that \( R(0, 0) = 0 \) and \( R(\infty, 0) = \infty \). Thus, for any \( b \in (0, \infty) \), there exists a unique \( q_A \in (0, \infty) \) that solves \( b = R(q_A, 0) \). The solution increases in \( b \). For this solution to induce the SE, it must satisfy \( q_A \leq q_a \), which is equivalent to \( b \leq B_a = R(q_a, 0) \). In addition, as proven below, \( J_2 > J_1 \) and \( p_0 < p_2 = p_1 \) in the SE. Also, \( V_1 = V_2 = V_0 \) in the SE. Thus, (2.13) and (2.11) are satisfied. Therefore, the SE exists if and only if \( b \leq B_a \).

To prove (iii) and (iv) in Proposition 3.1 for the SE, use \( p_1 = p_2 \), \( V_1 = V_0 \), and (3.7). Clearly, (2.13) is satisfied. With \( p_2 \) in (2.10), it is also clear that \( p_1 = p_2 \) satisfies (2.11). Because (2.9) and (3.7) imply \( (1 + r) J_1 - J_0 = \lambda [(1 + r) J_2 - J_0] \), then \( J_2 > J_1 \). Because \( V_1 = V_0 \) and \( q_0 > m(q_0) \), (2.4) implies

\[
p_0 = u_c - \frac{q_0}{m(q_0)} (1 + r - \sigma) V_0 < u_H - (1 + r - \sigma) V_0 = p_2.
\]

The PME: (2.14) yields \( p_1 = p(q_1) \) where

\[
p(q_1) = u_c + \lambda \sigma (V_1 - V_0) - \frac{q_1 (1 + r - \sigma) V_0}{(1 - \lambda) m(q_1)}. \tag{A.7}
\]
Also, \( q_1 = h^{-1}(q_0) \). Because a related buyer’s value \( V_1 \) satisfies \((1 + r)V_1 - \sigma V_0 = u_H - p_1 \), substituting \( p_1 \) from (A.7) and \((1 + r - \sigma)V_0 \) from (2.6) yields:

\[
V_1 - V_0 = \frac{1}{1 + \sigma} \left[ (1 - \lambda) (u_H - u_L) - m_0' \Delta_0 + \frac{q_1 m_0' (1 - \lambda)}{1 - \sigma} \right], \tag{A.8}
\]

where \( \hat{\sigma} = \frac{\sigma}{1 + r} \). As in the SE, \( J_2 = \frac{p_2 - c + (1 - \sigma) p_0}{1 + r - \sigma} \), and \((p_2, J_0) \) satisfy (A.3). In contrast, \( J_1 \) obeys (3.3) instead of (3.7). Substituting \( J_2 \) and \( rJ_0 \), I get

\[
J_1 - J_0 = \frac{1}{1 + \sigma} \left\{ \hat{\sigma} \lambda (u_H - c - m_0' \Delta_0) - \left[ \lambda (m_0 - q_0 m_0') \Delta_0 + (1 - \lambda)A \Delta_0 \right] \right\}, \tag{A.9}
\]

Substituting (A.8), (A.9) and the assumption \( u_L = c \) into the definition of \( \Delta_0 \) solves

\[
\Delta_0 = \frac{\hat{\lambda}(u_H - u_L) / \hat{\lambda}}{1 - \lambda \lambda_0 + (1 - \lambda)(m_0 - q_0 m_0') + (1 - \lambda)\hat{\lambda} \Delta_0 \Delta_0}, \tag{A.10}
\]

The PME exists if and only if (3.6) is satisfied. For \( q_1 > 0 \), a necessary (but not sufficient) condition is \( q_0 > h(0) \). Suppose \( q_0 > h(0) \). Substitute \((J_1 - J_0), \Delta_0 \) and \( q_1 m_0' (1 - \lambda) m_1 \) into (3.6) and impose \( u_L = c \) as assumed in the proposition. Then, (3.6) can be written as \( g_0(q_1, h(q_1)) < 0 \), where \( g_0 \) is defined by (A.2). By Lemma A.2, there exists a unique \( q_0 \), with \( h(0) < q_0 < q_0 \), such that (3.6) is satisfied if and only if \( q_0 > q_0 \).

In the PME, \( q_1 = h^{-1}(q_0) > 0 \), and \( q_0 \) solve \( b = R(q_0, h^{-1}(q_0)) \), where \( R \) is defined in (3.8). For any given \( q_1 \geq 0 \), the numerator of \( R(q_0, q_1) \) strictly increases in \( q_0 \) and the denominator of \( R(q_0, q_1) \) strictly decreases in \( q_0 \). Thus, \( R'(q_0, q_1) > 0 \). For any given \( q_0 > 0 \), the denominator of \( R(q_0, q_1) \) strictly decreases in \( q_1 \). The numerator of \( R(q_0, q_1) \) increases in \( q_1 \) if and only if the expression, \( \left[ q_1 + \frac{1 - \sigma(1 - \lambda) m(q_1)}{\sigma m(q_0)/q_0} \right] \), increases in \( q_1 \) for given \( q_0 \). I can verify that the partial derivative of this expression with respect to \( q_1 \) is positive, using \( \frac{m(q_0)}{q_0} > (1 - \lambda) \frac{m(q_1)}{q_1} > m'(q_1) \) when \( q_0 = h(q_1) \) (see Lemma A.1). Thus, \( R'(q_0, q_1) > 0 \). Because \( h(q) \) increases in \( q \), then \( \frac{d}{dq_0} R(q_0, h^{-1}(q_0)) > 0 \). Moreover, \( R(0, 0) = 0 \) and \( R(\infty, h^{-1}(\infty)) = \infty \). Therefore, for any given \( b \in (0, \infty) \), there exists a unique \( q_B > 0 \) that solves \( b = R(q_B, h^{-1}(q_B)) \). The solution increases in \( b \). For this solution to induce the PME, it must satisfy \( q_B > q_0 \), which is equivalent to \( b > B_B = R(q_B, h^{-1}(q_0)) \).
In addition, as proven below, \( J_2 > J_1 \) and \( p_1 < p_0 < p_2 \) in the PME. Thus, (2.13) is satisfied. If (2.11) is also satisfied, then the PME exists if and only if \( b > B_b \).

Notice that \( R(q, h^{-1}(q)) \geq R(q, 0) \) for all \( q \geq 0 \), where the inequality is strict if \( q > h(0) \). This implies \( R(q_B, 0) \leq R(q_B, h^{-1}(q_B)) = b = R(q_A, 0) \), where the inequality is strict if \( q_B > h(0) \). Thus, \( q_B \leq q_A \), with strict inequality if \( q_B > h(0) \).

Finally, I prove (iii) and (iv) in Proposition 3.1 for the PME. Since a type 0 buyer obtains the same expected surplus from visiting a type 1 and a type 0 seller, then

\[
(1 - \lambda) \frac{m(q_1)}{q_1} [u_e + \lambda \sigma(V_1 - V_0) - p_1] = \frac{m(q_0)}{q_0} [u_e + \lambda \sigma(V_1 - V_0) - p_0].
\]

Lemma A.1 shows that \( \mu_0(q_0) > (1 - \lambda) \frac{m(q_0)}{q_1} \) where \( q_0 = h(q_1) \). The above equation then implies \( p_1 < p_0 \). If \( p_0 < p_2 \), then \( p_1 < p_2 \), and so (2.13) is satisfied. The proofs of \( p_0 < p_2 \) and \( J_2 > J_1 \) are relegated to the Supplementary Appendix. QED

**B. Proof of Proposition 4.1**

The text preceding Proposition 4.1 has established that the social optimum should give full priority to repeat trades. For other parts of the proposition, let me examine the social planner’s maximization problem in (4.1). Using the adding-up constraint on the \( q \)'s to substitute \( q_0 \), I can verify that the planner’s objective function increases in \( q_1 \) if and only if \( (1 - \lambda)m'(q_1) > m'(q_0) \). Because \( m' \) is a strictly decreasing function and \( m'(0) = 1 \), the efficient choice of \( q_1 \) is at the corner \( q_1 = 0 \) if and only if \( 1 - \lambda \leq m'(q_0) \), i.e., if and only if \( q_0 \leq m^{-1}(1 - \lambda) \). If \( q_0 > m^{-1}(1 - \lambda) \), then the efficient choice of \( q_1 \) is positive and satisfies the first-order condition. Thus, the efficient choice of \( q_1 \) is

\[q_1 = h_e^{-1}(q_0) \text{ if } q_0 > m^{-1}(1 - \lambda), \text{ and } 0 \text{ otherwise,}\]

where \( h_e(q) \) is defined immediately before Proposition 4.1. Let me solve the efficient \( q_0 \). To do so, note that since the laws of motion of \( (\rho_1, \rho_2) \) in the efficient allocation are the same as in the equilibrium, then \( b = R(q_0, q_1) \) holds in both. If \( q_0 \leq m^{-1}(1 - \lambda) \), then \( q_1 = 0 \) and \( q_0 \) solves \( b = R(q_0, 0) \). The solution for \( q_0 \) exists and is unique. The solution satisfies
\[ q_0 \leq m'(1 - \lambda) \text{ if and only if } b \leq R(m'^{-1}(1 - \lambda), 0) = B_e. \] If \( b > B_e \), then \( q_1 = h_e^{-1}(q_0) > 0 \) and \( q_0 \) solves \( b = R(q_0, h_e^{-1}(q_0)) \). Again, the solution for \( q_0 \) exists and is unique.

Comparing the definition of \( h_e(q) \) and the definition of \( h(q) \) in (3.5), I get \( m'(h_e(q)) = \frac{m'(q)}{1 - x} < m'(h(q)) \) for all \( q \geq 0 \). Because \( m' \) is a strictly decreasing function, then \( h_e(q) > h(q) \) for all \( q \geq 0 \). To compare \( B_e \) with \( B_a \) and \( B_b \), note that

\[ m'^{-1}(1 - \lambda) = -\ln (1 - \lambda) < \ln \frac{2 - \lambda}{2(1 - \lambda)^2} = h(0) < q_b < q_a, \]

where the last two inequalities come from Lemma A.2. Because \( R(q_0, 0) \) increases in \( q_0 \) and \( h^{-1}(q_b) > 0 \), then \( B_e < R(q_a, 0) = B_a \) and \( B_e < R(q_b, 0) < R(q_b, h^{-1}(q_b)) = B_b \).

If \( b \leq B_e \), then \( b < B_a \), and the equilibrium is the SE. In this case, \( q_1 = 0 \) and \( b = R(q_0, 0) \) in both the social optimum and the equilibrium (SE). That is, the SE is socially efficient. If \( b > B_e \), the SE is inefficient because the social optimum requires partial mixing. To prove that the PME is also inefficient, note that the social optimum requires \( R(h_e(q_1), q_1) = b \) but the PME requires \( R(h(q_1), h(q_1)) = b \). Since \( h(q) < h_e(q) \) for all \( q \geq 0 \), then \( R(h(q_1), q_1) < R(h_e(q_1), q_1) \). Because \( R(h(q), q) \) and \( R(h_e(q), q) \) are both strictly increasing in \( q \), then \( q_1 \) in the PME is strictly higher than in the social optimum. Equivalently, I can express the result in terms of \( q_0 \) instead of \( q_1 \). Because \( h \) and \( h_e \) are strictly increasing functions, the result \( h(q) < h_e(q) \) for all \( q \geq 0 \) implies \( h^{-1}(q) > h_e^{-1}(q) \) for all \( q \geq 0 \). Thus, \( R(q, h^{-1}(q)) > R(q, h_e^{-1}(q)) \) for all \( q \geq 0 \). When \( b > B_e \), the efficient allocation requires \( R(q_0, h_e^{-1}(q_0)) = b \) and the PME requires \( R(q_0, h^{-1}(q_0)) = b \). Therefore, \( q_0 \) in the PME is strictly smaller than in the social optimum. \( \text{QED} \)
References


S. Supplementary Appendix

S.1. Proof of Lemma A.1

To prove (i), substitute $h(q)$ from (3.5). The inequality in (i) becomes $1 + \frac{\lambda}{(1-\lambda)m(q)} > 0$, which clearly holds.

For (ii), note that $t(q)$ is increasing and $t(q) \in \left[\frac{q}{2}, 1\right]$ for all $q \geq 0$. Then

$$h(q) = q + \ln\{a[1 + (a - 1)t(q)]\},$$

where $a \equiv \frac{1}{1-\lambda}$. Substituting $h(q)$, I can rewrite the inequality (ii) as $f(a, q) > 0$ where $f$ temporarily denotes

$$f(a, q) = a - \frac{m'(q)}{1 + (a - 1)t(q)} - \frac{m(q)}{q} \{q + \ln a + \ln [1 + (a - 1)t(q)]\}.$$

Compute $f(1, q) = 0$ and

$$f'(a, q) = 1 + \frac{t(q)m'(q)}{[1 + (a - 1)t(q)]^2} - \frac{m(q)}{q} \left[\frac{1}{a} + \frac{t(q)}{1 + (a - 1)t(q)}\right].$$

Then,

$$f'(1, q) = 1 - t(q) \left[\frac{m(q)}{q} - m'(q)\right] - \frac{m(q)}{q} > \frac{1}{q} \{q[1 + m'(q)] - 2m(q)\} > 0.$$

The first inequality uses the fact that $\frac{m(q)}{q} > m'(q)$ and $t(q) < 1$ for all $q \in (0, \infty)$.

The second inequality follows from verifying that $\{q[1 + m'(q)] - 2m(q)\}$ is an increasing function and that its value at $q = 0$ is 0. Moreover,

$$f''(a, q) = -\frac{2q^2(q)m''(q)}{[1 + (a - 1)t(q)]^4} + \frac{m'(q)}{q} \left[\frac{1}{a^2} + \frac{r^2(q)}{t(q)}\right] > m'(q) \left\{\frac{1}{a[1 + (a - 1)t(q)]} \left[\frac{1}{a} + \frac{r^2(q)}{t(q)}\right] + \frac{2(a-1)r^3(q)}{[1 + (a - 1)t(q)]^3}\right\} > 0.$$
The first equality comes from using $\frac{m(q)}{q} > m'(q)$ and re-arranging terms. The second inequality comes from $t(q) < 1$ and $a > 1$. Thus, $f'(a, q) > f'(1, q) > 0$ which, in turn, implies $f(a, q) > f(1, q) > 0$.

For (iii), the inequality $t'(q) \leq t(q)$ can be verified directly by using $m(q) = 1 - e^{-q}$. Then,

$$h'(q) = 1 + \lambda \left(\frac{m-qm'}{m^2}\right)' \left(1 - \lambda + \frac{m-qm'}{m^2} \right) \leq 1 + \frac{\lambda}{1-\lambda} \left(\frac{m-qm'}{m^2}\right) = (1 - \lambda) \frac{m'(q)}{m'(h(q))}.$$

This completes the proof of Lemma A.1. \textbf{QED}

S.2. Proof of Lemma A.2

Consider $g_a(\tilde{\eta}, h(\tilde{\eta}))$ where $g_a$ is defined by (A.1). Because $\tilde{\eta} = h^{-1}(q_0)$, $\tilde{\eta}$ maximizes $A(\tilde{\eta}, q_0)$ for any given $q_0$. Hence, the derivative of $A(\tilde{\eta}, q_0)$ with respect to $\tilde{\eta}$ is zero, and $\frac{d}{d\tilde{\eta}} g_a(\tilde{\eta}, h(\tilde{\eta})) = h'(\tilde{\eta}) \frac{d}{dq_0} g_a(\tilde{\eta}, q_0)$. Using this result and the fact $m'' = -m'$, I calculate:

$$\frac{m_0}{h'(\tilde{\eta})} \frac{d}{d\tilde{\eta}} g_a(\tilde{\eta}, h(\tilde{\eta})) = \tilde{g}_a(\tilde{\eta}) = m'_0 \left[ \frac{1 - \hat{\sigma}^2(1 - \lambda)}{1 - \hat{\sigma}(1 - \lambda)} - \hat{\sigma}m'(h(\tilde{\eta})) \right],$$

where I substituted $q_0 = h(\tilde{\eta})$. Lemma A.1 shows that $h'(q) \leq (1 - \lambda) \frac{m'_0}{m'_0}$. Then,

$$\tilde{g}_a'(\tilde{\eta}) = \frac{\hat{\sigma}m'_0}{1 - \hat{\sigma}(1 - \lambda)} h'(\tilde{\eta}) + \hat{\sigma}m'_0 h'(\tilde{\eta}) \leq \frac{-m'_0}{1 - \hat{\sigma}(1 - \lambda)} + \hat{\sigma}(1 - \lambda) m'_0 = m'_0 \left[ \hat{\sigma}(1 - \lambda) - \frac{1}{1 - \hat{\sigma}(1 - \lambda)} \right] < 0.$$

Moreover, $\tilde{g}_a(\infty) < 0$ and

$$\tilde{g}_a(0) = \frac{1 - \hat{\sigma}^2(1 - \lambda)}{1 - \hat{\sigma}(1 - \lambda)} - \hat{\sigma} \frac{2(1 - \lambda)^2}{2 - \lambda} > 1 - \frac{2(1 - \lambda)^2}{2 - \lambda} > 0.$$

Thus, $g_a(\tilde{\eta}, h(\tilde{\eta}))$ increases in $\tilde{\eta}$ when $\tilde{\eta}$ is small and decreases in $\tilde{\eta}$ when $\tilde{\eta}$ is large.

Compute:

$$g_a(0, h(0)) = \frac{m_0}{m_0} - \lambda + \hat{\sigma}(1 - \lambda) \left( \frac{m_0}{m_0} - q_0 \right) - \frac{\lambda}{1 - \hat{\sigma}(1 - \lambda)} \left( \frac{1}{m_0} - \frac{1}{1 - \lambda} \right) \left( \frac{m_0}{m_0} - q_0 \right) > 0,$$

where the first inequality follows from $\lambda < 1 - \hat{\sigma}(1 - \lambda)$, and the second inequality from $m_0/m'_0 > q_0$. It can be verified that $g_a(\infty, h(\infty)) = -\infty$. These properties of $g_a$ at the two endpoints and the properties of $dg_a/d\tilde{\eta}$ imply that there exists a unique $\tilde{\eta}_{1a} > 0$ such that
Substituting this result yields:

Then

imizes both types of dependence into account, I calculate:

The inequalities are evident. Also, it can be veriﬁed that

Now consider \( g_b(q_1, h(q_1)) \) where \( g_b \) is defined by (A.2). I prove that there is a unique \( q_{1b} > 0 \) such that \( g_b(q_{1b}, h(q_{1b})) = 0 \). Moreover, \( g_b < 0 \) if and only if \( q_1 > q_{1b} \), i.e., iff \( q_0 > q_b = h(q_{1b}) \). Because \( q_{1b} > 0 \), then \( g_b = h(q_{1b}) > h(0) \). Moreover, since \( g_b(q, h(q)) < g_a(q, h(q)) \) for all \( q \in [0, \infty) \), then \( q_{1b} < q_{1a} \) and \( q_b < q_a \).

First, I establish the properties of the derivative of \( g_b \). Because \( q_1 = h^{-1}(q_0) \), \( q_1 \) maximizes \( A(q_1, q_0) \) for any given \( q_0 \). Hence, the derivative of \( A(q_1, q_0) \) with respect to \( q_1 \) is zero. In contrast to the SE, \( g_b \) depends on \( q_1 \) separately from its dependence on \( q_0 \). Taking both types of dependence into account, I calculate:

Compute:

Then,

Substituting this result yields:

Using the fact \( m'' = -m' \), I compute:

The first inequality follows from an earlier result \( h' < (1 - \lambda) m'_1 / m'_0 \), and the last two inequalities are evident. Also, it can be verified that \( \hat{g}_b(q_1) / m'_1 \) is a strictly decreasing function.
Next, I compute the value of $\tilde{g}_b$ at the two endpoints. It is easy to verify that $\tilde{g}_b(\infty) = \frac{1 - \hat{\sigma}}{1 - \hat{\sigma}(1 - \lambda)} < 0$. For $q_1 = 0$, I have

$$
\tilde{g}_b(0) = \frac{1 - \hat{\sigma}}{1 - \hat{\sigma}(1 - \lambda)} + \hat{\sigma} (1 - m_0') - 3\hat{\sigma},
$$

which is ambiguous. If $\tilde{g}_b(0) > 0$, then as $q_1$ increases, $\tilde{g}_b$ is positive first and then becomes negative. In this case, $g_b(q_1, h(q_1))$ first increases and then decreases. If $\tilde{g}_b(0) \leq 0$, then as $q_1$ increases, $\tilde{g}_b$ becomes negative. In this case, $g_b(q_1, h(q_1))$ decreases in $q_1$ for all $q_1 > 0$.

Finally, I compute the value of $g_b$ at the two endpoints. It is easy to verify that $g_b(\infty, h(\infty)) < 0$. For $q_1 = 0$, I have

$$
g_b(0, h(0)) = \frac{\hat{\sigma}(1 - \lambda)}{m_0'} \left\{ \frac{1}{2 - \lambda} \left[ \frac{1 - 2\lambda(1 - \lambda)}{\hat{\sigma}} - \lambda + \frac{1 - \hat{\sigma}}{\hat{\sigma} \left[ 1 - \hat{\sigma}(1 - \lambda) \right]} \right] \right\} + (m_0 - q_0m_0'),
$$

where $q_0 = h(0)$. The expression in $[\cdot]$ is strictly decreasing in $\hat{\sigma}$. Its value at $\hat{\sigma} = 1$ is $(1 - 3\lambda + 2\lambda^2)$. Thus,

$$
\tilde{g}_b(0, h(0)) > \frac{\hat{\sigma}(1 - \lambda)}{m_0'} \left\{ \frac{1 - 3\lambda + 2\lambda^2}{2 - \lambda} + (m_0 - q_0m_0') \right\}
\quad = \frac{\hat{\sigma}(1 - \lambda)}{m_0'(2 - \lambda)} \left[ 1 - 2(1 - \lambda)^2 \ln \left( \frac{2 - \lambda}{2(1 - \lambda)^2} \right) \right] > 0.
$$

The equality follows from substituting $q_0$ and the last inequality from direct verification. If $\tilde{g}_b(0) > 0$, then $g_b(q_1, h(q_1))$ first increases from a positive value and then decreases to negative values; if $\tilde{g}_b(0) \leq 0$, then $g_b(q_1, h(q_1))$ decreases from a positive value to negative values monotonically. In both cases, there is a unique level $q_{1b} > 0$ such that $g_b(q_{1b}, h(q_{1b})) = 0$. Denote $q_b = h(q_{1b})$. Then, $g_b(q_1, h(q_1)) < 0$ if and only if $q_1 > q_{1b}$. \textbf{QED}

\textbf{S.3. Proofs of $p_0 < p_2$ and $J_2 > J_1$ in the PME}

To prove $J_2 > J_1$ in the PME, use $J_1$ in (A.9) and $\Delta_0$ in (A.10) (with $u_L = e$) to get

$$
\frac{\lambda}{\Delta_0} (1 + r) (1 + \lambda \hat{\sigma}) (J_2 - J_1)
= 1 - \lambda m_0' + \hat{\sigma} (1 - \lambda) (m_0 - q_0m_0') - A (q_1, q_0) \left[ \lambda + \frac{(1 - \lambda) \hat{\sigma} m_1}{\lambda + (1 - \lambda)m_1} \right].
$$

To prove that this expression is positive, temporarily denote it as $f_1(q_0, q_1, \hat{\sigma})$. Temporarily denote $a = \frac{1}{1 - \lambda}$. Calculate:

$$
f_1' = 1 + q_1 + (a - 1) t (q_1) - \frac{1}{a} (1 + q_0),
$$

4
$m = 1 - m'$ and $m'_1 = m'_0 a [1 + (a - 1) t(q_1)]$. Because $\ln (1 + x) < x$ for all $x > 0$, the definition of $h(q)$ implies

$$ q_0 = h(q_1) < q_1 + (a - 1) t(q_1) + \ln a. $$

Substituting this inequality for $q_0$, I get

$$(a) \ f'_1 \sigma = (a - 1) [1 + q_1 + (a - 1) t (q_1)] - \ln a.$$  

The derivative of this expression with respect to $a$ is positive for all $a \geq 1$. Because the value of this expression at $a = 1$ is 0, then $(a) f'_1 \sigma \geq 0$ for all $a \geq 1$. Thus, 

$$f_1(q_0, q_1, \sigma) > f_1(q_0, q_1, 0) = 1 - \lambda m'_0 - \lambda A(q_1, q_0).$$

Since $q_1 > 1$, the definition of $A$ implies

$$A < [\lambda + (1 - \lambda) m_1] (1 - m'_0) < [\lambda + (1 - \lambda) m_1] (1 - \lambda m'_0).$$

Then

$$1 - \lambda m'_0 - A(q_1, q_0) > (1 - \lambda) (1 - m_1) (1 - \lambda m'_0) > 0.$$  

This implies $f_1(q_0, q_1, 0) > 0$. Hence, $f_1(q_0, q_1, \sigma) > 0$ and $J_2 > J_1$. 

To prove $p_0 < p_2$, I first prove the following results in the PME:

$$(a) \ \frac{A(q_1, q_0)}{(1-\lambda)m_1} > \frac{\lambda[1-\lambda m'_0+(1-\lambda)(m_0-q_0 m'_0)]}{\lambda+1-\lambda^2};$$

$$(b) \ \frac{m'_0}{m_0} > m'_1 + m_0 - q_0 m'_0.$$ 

For $(a)$, recall that the existence condition for the PME is $g_0(q_1, q_0) < 0$, where $g_0$ is given by (A.2). The condition $g_0 < 0$ is equivalent to

$$A(q_1, q_0) > \frac{1 - \lambda m'_0 + \sigma (1 - \lambda) (m_0 - q_0 m'_0)}{\frac{\lambda + (1 - \lambda) m_1}{1 - \sigma (1 - \lambda)}} = \frac{1 - \lambda m'_0 + \sigma (1 - \lambda) (m_0 - q_0 m'_0)}{\frac{\lambda + (1 - \lambda) m_1}{1 - \sigma (1 - \lambda)}}.$$  

Given $(q_0, q_1)$, the derivative of the right-hand side with respect to $\sigma$ has the same sign as

$$(1 - \lambda m'_0) \left[ m_1 + \frac{\lambda + (1 - \lambda) m_1}{(1 - \sigma (1 - \lambda))} \right] + \frac{\lambda + (1 - \lambda) m_1}{1 - \sigma (1 - \lambda)} (m_0 - q_0 m'_0) \left[ 1 - \frac{\sigma (1 - \lambda)}{1 - \sigma (1 - \lambda)} \right]$$

$$(1 - \lambda m'_0) \left[ m_1 + \frac{\lambda + (1 - \lambda) m_1}{(1 - \sigma (1 - \lambda))} \right] + \frac{\lambda + (1 - \lambda) m_1}{1 - \sigma (1 - \lambda)} (m_0 - q_0 m'_0)$$

$$(1 - \lambda m'_0) m_1 - \frac{\lambda + (1 - \lambda) m_1}{1 - \sigma (1 - \lambda)} m'_0 (1 - \lambda + q_0) < 0.$$
Since \( \hat{\sigma} < 1 \), then the lower bound on \( \frac{A(q_1, q_0)}{\lambda + (1 - \lambda) m_1} \) above is greater than its value at \( \hat{\sigma} = 1 \); that is,

\[
\frac{A(q_1, q_0)}{(1 - \lambda) m_1} > \frac{\lambda [1 - \lambda m'_0 + (1 - \lambda) (m_0 - q_0 m'_0)]}{\lambda + (1 - \lambda^2) m_1}.
\]

Because \( m_1 < 1 \), this bound is larger than the one in (a).

For (b), substitute \( \Delta_0 \) from (A.10) (with \( u_L = c \)) to compute

\[
\frac{\lambda(u_H - c)}{\Delta_0} = 1 - \hat{\sigma} + \hat{\sigma} \lambda m'_0 + \hat{\sigma} [1 - \hat{\sigma} (1 - \lambda)] (m_0 - q_0 m'_0) - \frac{(1 - \lambda) m_1 A \hat{\sigma} (1 - \hat{\sigma})}{\lambda + (1 - \lambda) m_1}.
\]

Given \((q_0, q_1)\), compute the derivative:

\[
\frac{\partial}{\partial \sigma} \left[ \frac{\lambda(u_H - c)}{\Delta_0} \right] = -1 + \lambda m'_0 + \left[ 1 - 2\hat{\sigma} (1 - \lambda) \right] (m_0 - q_0 m'_0) - \frac{(1 - \lambda) m_1 A (1 - 2\hat{\sigma})}{\lambda + (1 - \lambda) m_1}.
\]

If \( \hat{\sigma} \leq 1/2 \), this derivative is increasing in \( \lambda \) for given \((q_0, q_1)\), and so its value is smaller than the value at \( \lambda = 1 \), which is \(-q_0 m'_0 < 0\). If \( \hat{\sigma} > 1/2 \), then

\[
\frac{\partial}{\partial \sigma} \left[ \frac{\lambda(u_H - c)}{\Delta_0} \right] < -1 + \lambda m'_0 + \lambda (m_0 - q_0 m'_0) + \frac{(1 - \lambda) m_1 A}{\lambda + (1 - \lambda) m_1} - \lambda q_0 m'_0 < 0.
\]

Thus, for any given \((q_0, q_1)\), \( \frac{\lambda(u_H - c)}{\Delta_0} \) is larger than its value at \( \hat{\sigma} = 1 \), which establishes (b).

Now I prove \( p_0 < p_2 \) in the PME. Substituting \( p_0 \) and \( p_2 \) yields:

\[
\frac{1 + \lambda \hat{\sigma}}{\Delta_0} (p_2 - p_0) = (1 - \lambda) \frac{u_H - c}{\Delta_0} m'_0 \left[ (1 + \lambda \hat{\sigma}) \left( \frac{q_0}{m_0} - 1 \right) - \lambda \hat{\sigma} \left( \frac{q_1}{(1 - \lambda) m_1} - 1 \right) \right]
\]

\[
> (1 - \lambda) \frac{u_H - c}{\Delta_0} m'_0 \left[ (1 + \lambda) \left( \frac{q_0}{m_0} - 1 \right) - \lambda \left( \frac{q_1}{(1 - \lambda) m_1} - 1 \right) \right]
\]

\[
= (1 - \lambda) \frac{u_H - c}{\Delta_0} m'_0 + (1 + \lambda) \left( \frac{q_0}{m_0} - 1 \right) (m_0 - q_0 m'_0) + \lambda \left( 1 - m'_0 \right) + \frac{\lambda A}{\lambda + (1 - \lambda) m_1}.
\]

The inequality follows from the fact \( \frac{m_0}{q_0} > \frac{(1 - \lambda) m_1}{q_1} \) (see Lemma A.1), which implies that the expression is decreasing in \( \hat{\sigma} \) for given \((q_0, q_1)\). The last equality comes from using the definition of \( A \) to substitute \( \frac{q_1}{(1 - \lambda) m_1} \) and re-arranging terms. Substituting (a) and (b) from the above, I get:

\[
\frac{1 + \lambda \hat{\sigma}}{\Delta_0} (p_2 - p_0) > (1 - \lambda) \frac{u_H - c}{\Delta_0} m'_0 + (1 + \lambda) \left( \frac{q_0}{m_0} - 1 \right) m'_0 - \lambda (1 - m'_0) + \frac{\lambda A}{\lambda + (1 - \lambda) m_1}
\]

\[
> (1 - \lambda - \frac{1 + \lambda}{m_0}) (m_0 - q_0 m'_0) + 1 - \lambda m'_0 + \frac{\lambda^2 [1 - \lambda (1 - \lambda) (m_0 - q_0 m'_0)]}{\lambda + 1 - \lambda^2}
\]

\[
= \frac{1 + \lambda}{1 + \lambda - \lambda^2} \left[ 1 - \lambda m'_0 + \left( 1 - \lambda - \frac{1 + \lambda - \lambda^2}{m_0} \right) (m_0 - q_0 m'_0) \right]
\]

\[
> m'_0 \left( \frac{q_0}{m_0} - 1 \right) > 0.
\]
The first inequality copies the previous result. The second inequality comes from substituting (a) and (b). The third inequality comes from the fact that the expression in [...] is strictly decreasing in λ for any given q0 and, hence, is greater than its values at λ = 1. The last inequality comes from \( \frac{q}{m(q)} > 1 \) for all \( q > 0 \). QED

S.4. Proof of Proposition 5.1

This proposition is for the SE. For (i), note that a seller who posts the high price \( p_2 \) can be a type 2 or a type 1 seller. The seller succeeds in trade with certainty if the seller is type 2, and with probability \( \lambda \) if the seller is type 1. In contrast, a type 0 seller, who posts the low price \( p_0 \), succeeds in trade with probability \( m(q_0) \). Since \( m(q_0) < 1 \) for all \( b < \infty \), then \( m(q_0) < \lambda \) for \( \lambda \) sufficiently close to 1.

For (ii) and (iii), use the steady state version of (3.1) to substitute \( \rho_0 \) and \( \rho_2 \). I get \( \rho_0 m(q_0) = \rho_1/\sigma \) and
\[
\rho_0 m(q_0) + \rho_1 \lambda + \rho_2 = \rho_1 \left( \frac{1}{\sigma} + \frac{\lambda}{1 - \sigma} \right).
\]
Since the frequency of a sale is the ratio between these two quantities, it is equal to \( \frac{1}{\sigma m(q_0)} \). Because \( q_0 \) is an increasing function of \( b \) (see Proposition 3.1), the duration of a sale decreases in \( b \).

To verify (iv), note that \( V_1 = V_0 \) in the SE. Substituting \( (1 + r - \sigma) V_0 \) from (2.6) and \( \Delta_0 \) from (A.5) into (2.4), and using the assumption \( u_L = c \), I obtain the price \( p_0 \). Then, the markup by a type 0 seller is:
\[
mk\rho_0 = \lambda \left( \frac{u_H}{c} - 1 \right) \left[ 1 - \frac{1}{G(q_0, \lambda)} \right],
\]
where \( G \) temporarily denotes
\[
G(q, \lambda) = \frac{m}{q} \left[ \frac{1 - \hat{\sigma}}{m'} + \lambda \hat{\sigma} + \hat{\sigma} \left[ 1 - \hat{\sigma} (1 - \lambda) \right] \left( \frac{m}{m'} - q \right) \right],
\]
with \( \hat{\sigma} = \sigma/(1 + r) \). Since \( m \geq qm' \) for all \( q \geq 0 \), then \( G(q, \lambda) > 0 \) for all \( q \geq 0 \). Thus, \( mkp\rho_0 < 0 \) if and only if \( G(q_0, \lambda) < 1 \). Also, since \( q_0 \) is determined by \( b = R(q_0, 0) \) which does not depend on \( u_H/c \), \( mkp\rho_0 \) decreases with \( u_H/c \) if and only if \( G(q_0, \lambda) < 1 \). Note that
\[ G(0, \lambda) = 1 - \hat{\sigma} (1 - \lambda) < 1 \text{ and } G(\infty, \lambda) = \infty. \] Below I will prove that there exists \( q_{c0} > 0 \) such that \( G(q_0, \lambda) < 1 \) if and only if \( q_0 < q_{c0} \). Moreover, if \( \lambda \leq \frac{1}{\sigma} - 1 \), then \( G'_{q_0}(q_0, \lambda) > 0 \) for all \( q_0 > 0 \). If \( \lambda > \frac{1}{\sigma} - 1 \), then there exists \( q_{c1} \in (0, q_{c0}) \) such that \( G'_{q_{c0}}(q_0, \lambda) < 0 \) if and only if \( q_0 < q_{c1} \). Define \( B_0 = R(q_{c0}, 0) \) and \( B_1 = R(q_{c1}, 0) \). Because \( b = R(q_0, 0) \) and \( \frac{db}{dq} > 0 \), the properties of \( mkp0 \) stated in (iv) follow.

To establish the properties of \( G \) stated above, compute:

\[
q^2 G'_q(q, \lambda) = - (m - qm') \left[ \frac{1}{m} - \lambda \hat{\sigma} + \hat{\sigma} [1 - \hat{\sigma} (1 - \lambda)] \left( \frac{m}{m'} - q \right) \right] + qm \left[ \frac{1}{m'} - \hat{\sigma} [1 - \hat{\sigma} (1 - \lambda)] \left( \frac{m}{m'} \right) \right],
\]

where I have used the fact \( m'' = -m' \). Further compute:

\[
\frac{1}{q} \left( q^2 G'_q \right)'_q = \frac{1}{m'} - \lambda \hat{\sigma} m' + \hat{\sigma} [1 - \hat{\sigma} (1 - \lambda)] \left( m + \frac{m}{m'} + qm' \right),
\]

where I have used the fact \( m' = 1 - m \). It is easy to verify that \( m + \frac{m}{m'} + qm' \) strictly increases in \( q \), and so does \( \left[ \frac{1}{q} \left( q^2 G'_q \right)'_q \right] \). At \( q = 0 \), the value of \( \left[ \frac{1}{q} \left( q^2 G'_q \right)'_q \right] \) is \( 1 - \hat{\sigma} - \hat{\sigma} \lambda \), which is positive if and only if \( \lambda < \frac{1}{\sigma} - 1 \). Also, since \( \frac{m - qm'}{q^2} \to \frac{1}{2} \) when \( q \to 0 \), then \( G'_q(0, \lambda) = \frac{1}{2} (1 - \hat{\sigma} - \hat{\sigma} \lambda) \), which is positive if and only \( \lambda < \frac{1}{\sigma} - 1 \).

If \( \lambda \leq \frac{1}{\sigma} - 1 \), then \( \frac{1}{q} \left( q^2 G'_q \right)'_q \to 0 \) and \( G'_q \to 0 \) for all \( q > 0 \). Since \( G(0, \lambda) < 1 \) and \( G(\infty, \lambda) = \infty \), there is a unique \( q_{c0} > 0 \) such that \( G(q, \lambda) < 1 \) if and only if \( q < q_{c0} \). If \( \lambda > \frac{1}{\sigma} - 1 \), then \( \left[ \frac{1}{q} \left( q^2 G'_q \right)'_q \right]_{q=0} < 0 \) and \( G'_q(0, \lambda) < 0 \). In this case, there exists \( q_{c2} > 0 \) such that \( \frac{1}{q} \left( q^2 G'_q \right)'_q < 0 \) if and only if \( q < q_{c2} \), and there exists \( q_{c1} > q_{c2} \) such that \( G'_q(q, \lambda) < 0 \) if and only if \( q < q_{c1} \). Because \( G(0, \lambda) < 1 \) and \( G(\infty, \lambda) = \infty \), there exists \( q_{c0} > q_{c1} \) such that \( G(q, \lambda) < 1 \) if and only if \( q < q_{c0} \). \( \text{QED} \)